Объединенный семинар ЦКП ССКЦ СО РАН и НГУ

Высокопроизводительные вычисления

Математическое моделирование термотоков в металле при импульсном воздействии на основе решения регуляризованного уравнения электростатики

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15 октября 2026 г.

Исследование выполнено за счет гранта Российского научного фонда (проект № 25-11-00154).

Цель работы

Анализ результатов экспериментов импульсного нагрева вольфрамовой пластинки и вращения расплава.

Расширение математической модели термотоков.

Модель полностью описывающая эксперимент.

Изучение эффектов кластеризации на основе дополнительной информации.

Возможность вычислений на существующих пакетах программ.

Observation of the Tungsten Surface Damage under ITER-relevant Transient Heat Loads during and after Electron Beam Pulse

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Abstract. Wide-area (2 cm^2) high-power (up to 7 MW) si of intense pulsed heat loads on tungsten sample. Various damage: IR imaging of the front view of the target, captur recording of intensity of thermal radiation from several sp near and above melting threshold. Formation of the cracl pulses was observed. Two-dimensional temperature distri and spalling of material were shown on severe damaged t 0.1 – 0.2 mm was revealed with transverse microsections. exposures.

Motion of Molten Layer

Multiple exposures with average heat load above the melting threshold (F_{HF}>100 MJm^{-2}s$^{-0.5}$) caused formation of helical structure in the picture of thermal radiation of the target surface. The structure was located near the border of irradiation area. Comparison of the successive NIR images 100 μs after beam ending revealed a circular motion of this structure (Fig. 5(a,b)). Electron current flows from edge to centre of the target and magnetic field is directed perpendicularly from the surface, so direction of this motion correlates with J×B force. A set of five subsequent pictures showed that helical structure on a radius of 5 mm moved on a distance of ~0.1 mm per one heating pulse.

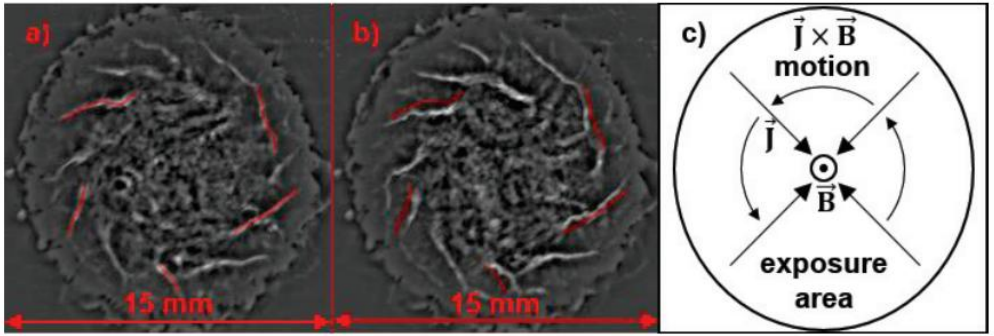
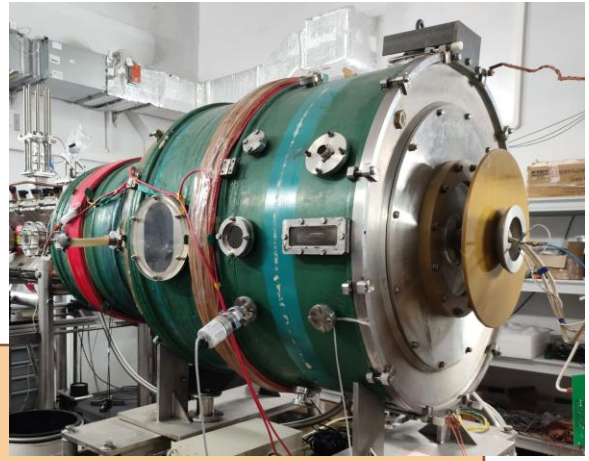
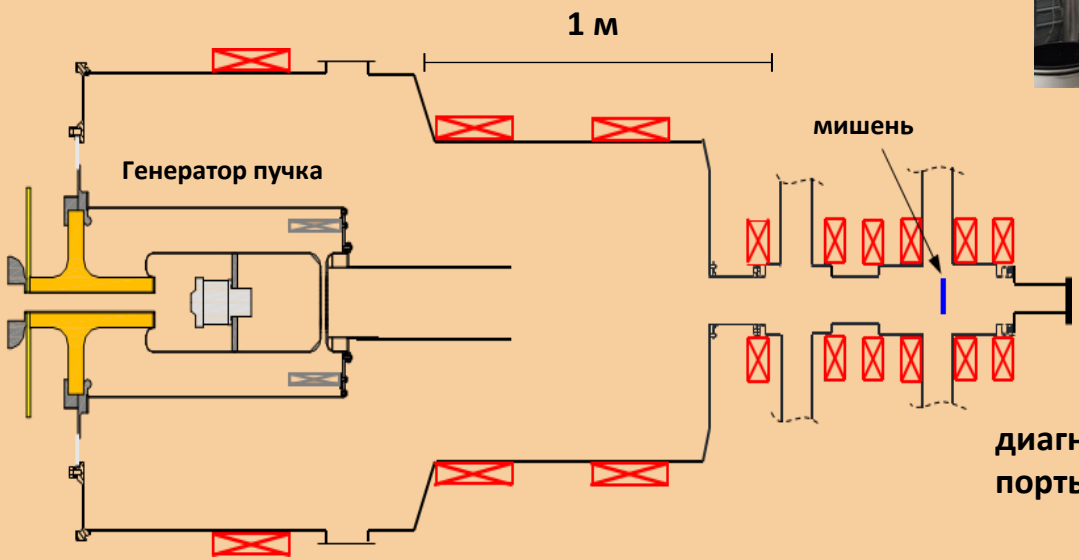
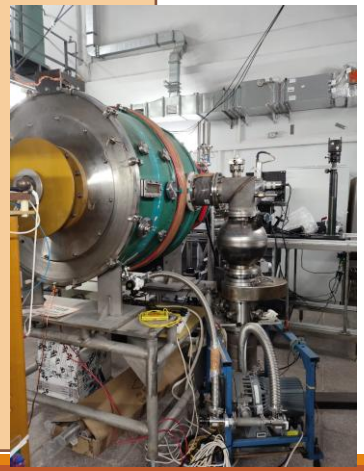


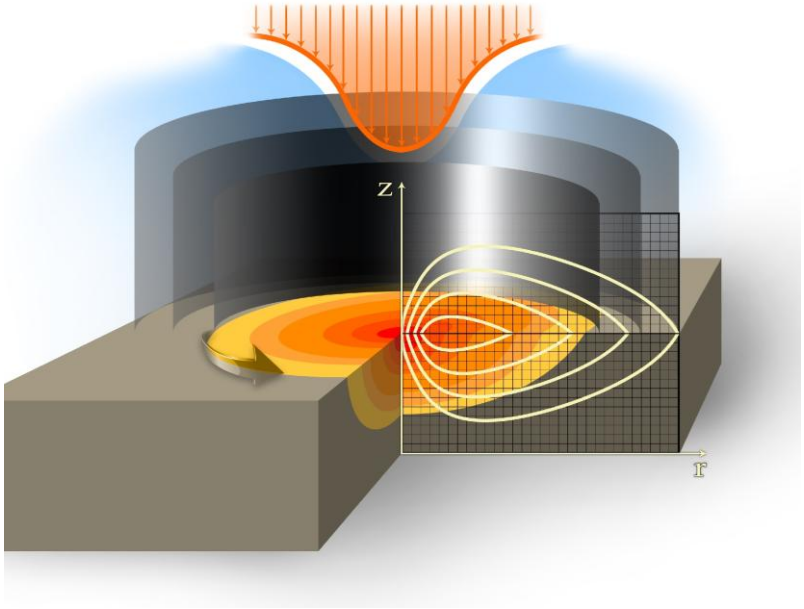
FIGURE 5. Picture of molten layer motion. a) the reference IR image obtained after heat load end; b) IR image captured after 4 successive exposures; c) directions of the magnetic field, current and J×B motion. Red curves on (b) mark inhomogeneities on (a)

BETA (Beam of Electrons for materials Test Applications) ИЯФ СО РАН









Электронный пучок с гауссовым профилем распределения мощности падает на поверхность вольфрама в ведущем магнитном поле.

На поверхности задано модельное нормальное распределение температуры. Распределение температуры в парах над образцом повторяет распределение температуры на поверхности. Вглубь образца распределение температуры задаётся линейным образом.

$$\frac{\partial}{\partial r}\left(A_r\frac{\partial J}{\partial r}\right)+\frac{\partial}{\partial z}\left(A_z\frac{\partial J}{\partial z}\right)=2\pi F(T,S)$$

$$J|_{(0,z)}=0,\ \left.\frac{\partial J}{\partial z}\right|_{(r_{\max},z)}=0,\ \left.\frac{\partial J}{\partial r}\right|_{(r,z_{\max})}=0,\ J|_{(r,0)}=0$$

Модель основана на модификации системы уравнений Максвелла в цилиндрической системе координат.

Задача рассматривается в двух измерениях: (r,z).

$$B \to 0$$

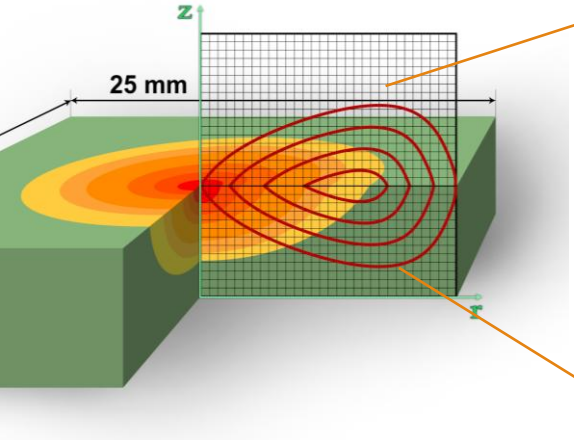
$$\rho_0 \frac{\partial^2 J}{\partial r^2} + \frac{\partial J}{\partial r}\left(\frac{\partial \rho_0}{\partial r} - \frac{\rho_0}{r}\right) + \rho_0 \frac{\partial^2 J}{\partial z^2} + \frac{\partial J}{\partial z}\frac{\partial \rho_0}{\partial z} = -2\pi r\left(\frac{\partial S_0}{\partial r}\frac{\partial T}{\partial z} - \frac{\partial T}{\partial r}\frac{\partial S_0}{\partial z}\right)$$

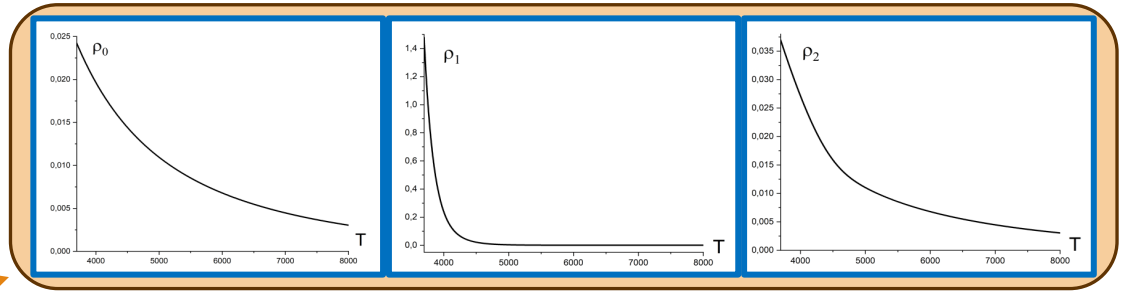
$$\rho_0 \frac{\partial^2 J}{\partial r^2} + \frac{\partial J}{\partial r}\left(\frac{\partial \rho_0}{\partial r} - \frac{\rho_0}{r}\right) + \rho_0(\Omega^2+1)\frac{\partial^2 J}{\partial z^2} + \left((\Omega^2+1)\frac{\partial \rho_0}{\partial z} + \rho_0 2\Omega\frac{\partial \Omega}{\partial z}\right)\frac{\partial J}{\partial z} =$$

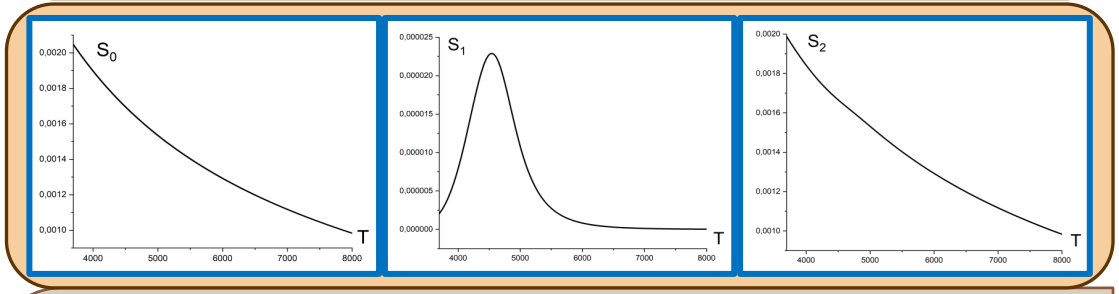
$$= -2\pi r\left(\frac{\partial S_0}{\partial r}\frac{\partial T}{\partial z} - \frac{\partial T}{\partial r}\frac{\partial S_0}{\partial z}\right)$$

$$\rho_0 \frac{\partial^2 J}{\partial r^2} + \frac{\partial J}{\partial r}\left(\frac{\partial \rho_0}{\partial r} - \frac{\rho_0}{r}\right) + \left(\frac{\rho_1^2}{\rho_2} + \rho_2\right)\frac{\partial^2 J}{\partial z^2} + \frac{\partial J}{\partial z}\left(-\frac{\rho_1^2}{\rho_2^2}\frac{\partial \rho_2}{\partial z} + \frac{\partial \rho_2}{\partial z} + \frac{2\rho_1}{\rho_2}\frac{\partial \rho_1}{\partial z}\right) =$$

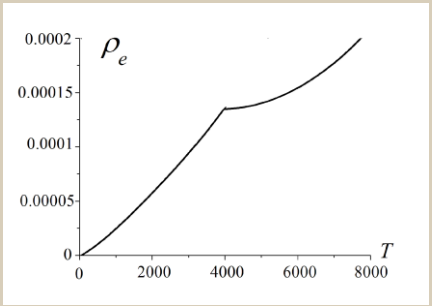
$$-2\pi r\left(-\frac{\rho_1}{\rho_2}\left(S_1\frac{\partial^2 T}{\partial r\,\partial z} + \frac{\partial T}{\partial r}\frac{\partial S_1}{\partial z}\right) + (S_0 - S_2)\frac{\partial^2 T}{\partial r\,\partial z} - \frac{S_1}{\rho_2}\frac{\partial T}{\partial r}\frac{\partial \rho_1}{\partial z} + \frac{\rho_1 S_1}{\rho_2^2}\frac{\partial \rho_2}{\partial z}\frac{\partial T}{\partial r} + \left(\frac{\partial S_0}{\partial r}\frac{\partial T}{\partial z} - \frac{\partial T}{\partial r}\frac{\partial S_2}{\partial z}\right)\right)$$

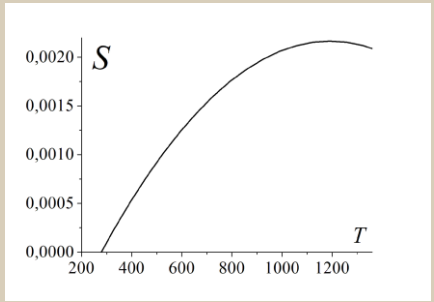






Экспериментальные данные





Термоэлектрические коэффициенты в газовой фазе

$$\rho = (\sigma)^{-1}$$

$$\sigma = \frac{4}{3\sqrt{\pi}} \frac{q^2 n_e}{m} \int_0^\infty u^{3/2} \tau e^{-u} \begin{pmatrix} \frac{1}{1+\Omega^2} & -\frac{\Omega}{1+\Omega^2} & 0 \\ \frac{\Omega}{1+\Omega^2} & \frac{1}{1+\Omega^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{d}u$$

$$K = \sigma S = -\frac{4}{3\sqrt{\pi}} \frac{q n_e}{m} \int_0^\infty (\eta + u) u^{3/2} \tau e^{-u} \begin{pmatrix} \frac{1}{1+\Omega^2} & -\frac{\Omega}{1+\Omega^2} & 0 \\ \frac{\Omega}{1+\Omega^2} & \frac{1}{1+\Omega^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{d}u$$

$\vec{\Omega} = \frac{q\tau \vec{B}}{m}$ Ларморовская частота

$u = \mathcal{E}/T$ кинетическая энергия электрона

$\eta = -\tilde{\varphi}/T = -\ln\left(\frac{n_e}{2}\left(\frac{mT}{2\pi\hbar^2}\right)^{-3/2}\right) > 0$ электрохимическая энергия

$$\begin{pmatrix} A_2 & -A_1 & 0 \\ A_1 & A_2 & 0 \\ 0 & 0 & A_0 \end{pmatrix}$$

$\mathcal{E}$ относительная энергия электрона

$\tilde{\varphi}$ электрохимический потенциал

V. A. Popov, G. G. Lazareva Evaluation of the Influence of Magnetic Field on the Magnitude of Thermal Currents During Pulsed Heating of Tungsten by an Electron Beam // Lobachevskii Journal of Mathematics, Vol. 46, No. 9, pp. 4465-4474 (2025).

Система уравнений Максвелла

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j}, \quad \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{j} = 0,$$

$$\mathbf{j} = \nabla \times \mathbf{F}$$

$$\mathbf{j} = \begin{pmatrix} j_r \\ j_\varphi \\ j_z \end{pmatrix} = \begin{pmatrix} \frac{1}{r}\frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z} \\ \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \\ \frac{1}{r}\frac{\partial\ F_\varphi}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} -\frac{\partial F_\varphi}{\partial z} \\ 0 \\ \frac{1}{r}\frac{\partial\ F_\varphi}{\partial r} \end{pmatrix}$$

$\mathbf{j} = (j_r,\ j_\varphi,\ j_z)$ - ток

$\mathbf{F} = (F_r,\ F_\varphi,\ F_z)$

векторный потенциал тока

Обобщённый закон Ома

$$\vec{j} = (\vec{E} - S\nabla T - \nabla\mu/e)/\rho_e,$$

$$\nabla \times (\rho_e \vec{j}) = \nabla \times (\vec{E} - S\nabla T - \nabla\mu/e) = \nabla S \times \nabla T.$$

ρ_e - удельное сопротивление

S - термоЭДС

μ - химический потенциал электронов

$J = 2\pi r F_\varphi$ - полоидальный ток

$$\frac{\partial^2 J}{\partial r^2}\rho_0 + \frac{\partial J}{\partial r}\left(\frac{\partial \rho_0}{\partial r} - \frac{\rho_0}{r}\right) + \frac{\partial^2 J}{\partial z^2}\left(\frac{\rho_1^2}{\rho_2} + \rho_2\right) + \frac{\partial J}{\partial z}\left(-\frac{\rho_1^2}{\rho_2^2}\frac{\partial \rho_2}{\partial z} + \frac{\partial \rho_2}{\partial z} + \frac{2\rho_1}{\rho_2}\frac{\partial \rho_1}{\partial z}\right)$$
$$= 2\pi r\left(\frac{\partial^2 T}{\partial r \partial z}\left(S_0 - S_2 - S_1\frac{\rho_1}{\rho_2}\right) + \frac{\partial T}{\partial r}\left(\frac{\rho_1 S_1}{\rho_2^2}\frac{\partial \rho_2}{\partial z} - \frac{S_1}{\rho_2}\frac{\partial \rho_1}{\partial z} - \frac{\rho_1}{\rho_2}\frac{\partial S_1}{\partial z} - \frac{\partial S_2}{\partial z}\right) + \frac{\partial S_0}{\partial r}\frac{\partial T}{\partial z}\right)$$

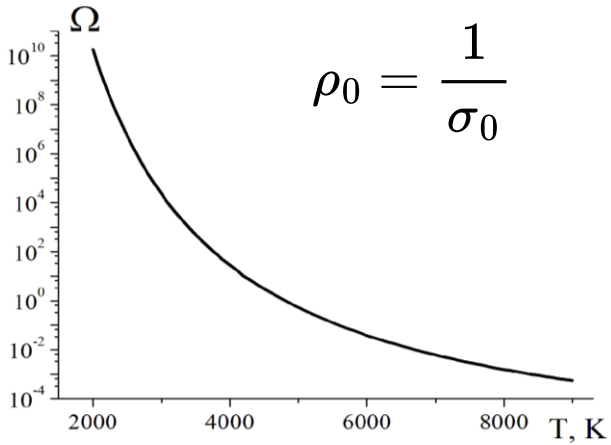
Упрощённая модель:

Ларморовская частота определена при постоянном относительно энергии электрона времени свободного пробега.

$$\sigma = \frac{\tau q^2 n_e}{m} \begin{pmatrix} \frac{1}{1+\Omega^2} & -\frac{\Omega}{1+\Omega^2} & 0 \\ \frac{\Omega}{1+\Omega^2} & \frac{1}{1+\Omega^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \sigma_0 \begin{pmatrix} \frac{1}{1+\Omega^2} & -\frac{\Omega}{1+\Omega^2} & 0 \\ \frac{\Omega}{1+\Omega^2} & \frac{1}{1+\Omega^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho = (\sigma)^{-1}$$

$$K = \sigma S = -\frac{\tau\left(\eta + \frac{5}{2}\right) q n_e}{m} \begin{pmatrix} \frac{1}{1+\Omega^2} & -\frac{\Omega}{1+\Omega^2} & 0 \\ \frac{\Omega}{1+\Omega^2} & \frac{1}{1+\Omega^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\rho_0 = \frac{1}{\sigma_0}$$

$$\rho_0 \frac{\partial^2 J}{\partial r^2} + \frac{\partial J}{\partial r}\left(\frac{\partial \rho_0}{\partial r} - \frac{\rho_0}{r}\right) + \rho_0(\Omega^2+1)\frac{\partial^2 J}{\partial z^2} + \left((\Omega^2+1)\frac{\partial \rho_0}{\partial z} + \rho_0 2\Omega \frac{\partial \Omega}{\partial z}\right)\frac{\partial J}{\partial z} =$$

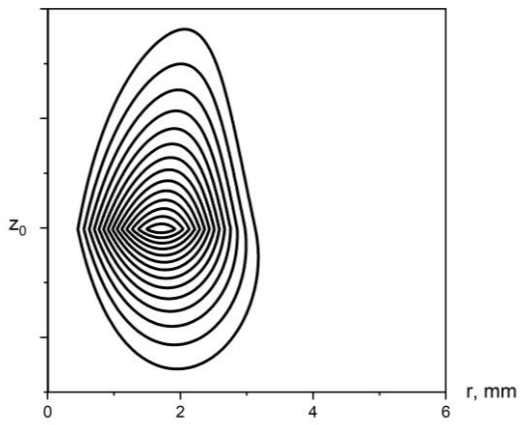
$$= -2\pi r\left(\frac{\partial S_0}{\partial r}\frac{\partial T}{\partial z} - \frac{\partial T}{\partial r}\frac{\partial S_0}{\partial z}\right)$$

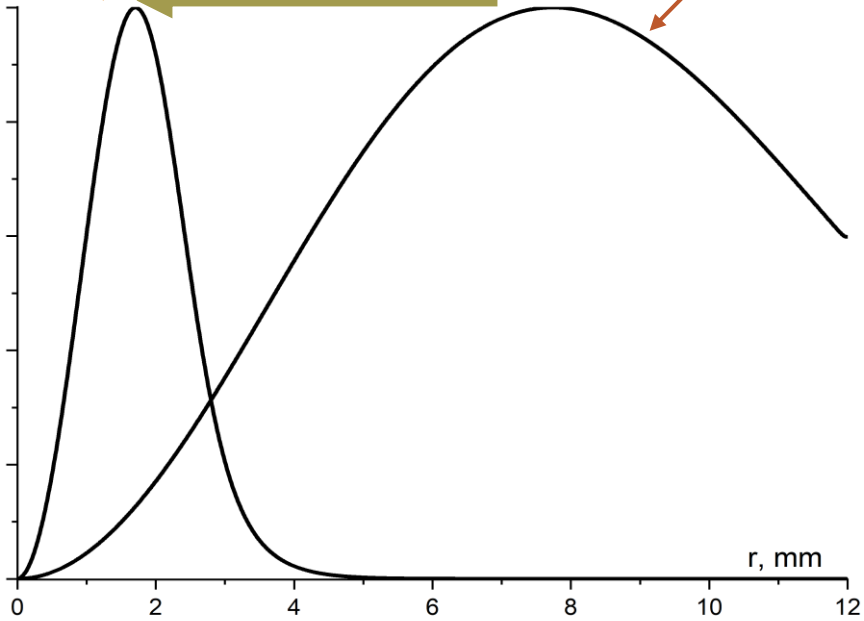
$$\rho_0\frac{\partial^2 J}{\partial r^2}+\left(\frac{\partial \rho_0}{\partial r}-\frac{\rho_0}{r}\right)\frac{\partial J}{\partial r}+\rho_0(\Omega^2+1)\frac{\partial^2 J}{\partial z^2}+$$

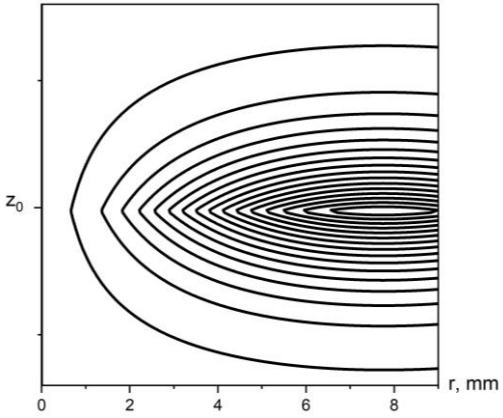
$$+\left((\Omega^2+1)\frac{\partial \rho_0}{\partial r}+\rho_0 2\Omega\frac{\partial \Omega}{\partial z}\right)\frac{\partial J}{\partial z}=-2\pi r\left(\frac{\partial S_0}{\partial r}\frac{\partial T}{\partial z}-\frac{\partial T}{\partial r}\frac{\partial S_0}{\partial z}\right)$$

$B \to 0$

$$\rho_0\frac{\partial^2 J}{\partial r^2}+\frac{\partial J}{\partial r}\left(\frac{\partial \rho_0}{\partial r}-\frac{\rho_0}{r}\right)+\rho_0\frac{\partial^2 J}{\partial z^2}+\frac{\partial J}{\partial z}\frac{\partial \rho_0}{\partial z}=-2\pi r\left(\frac{\partial S_0}{\partial r}\frac{\partial T}{\partial z}-\frac{\partial T}{\partial r}\frac{\partial S_0}{\partial z}\right)$$



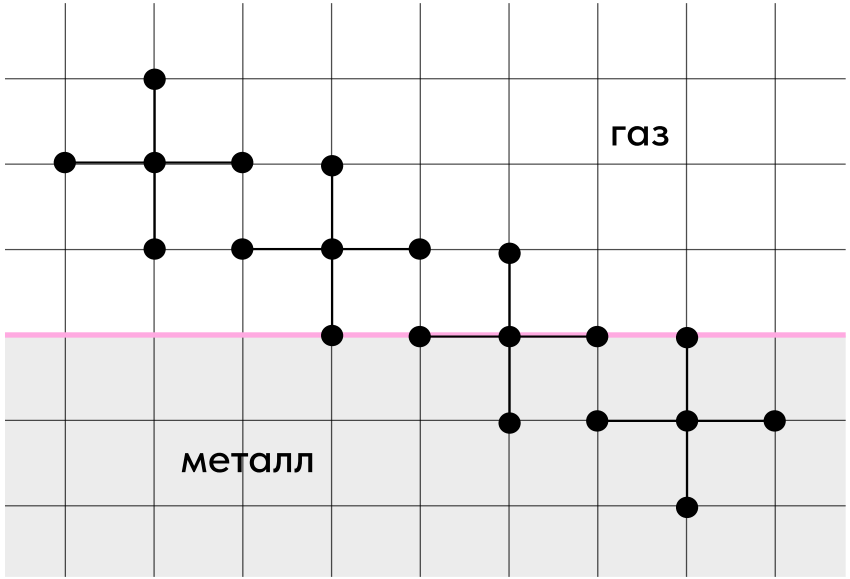




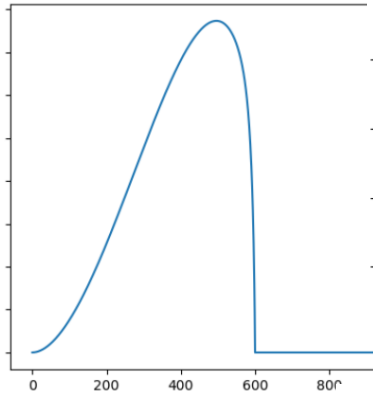
$$\rho_0 \frac{\partial^2 J}{\partial r^2} + \left(\frac{\partial \rho_0}{\partial r} - \frac{\rho_0}{r} \right) \frac{\partial J}{\partial r} + \rho_0 (\Omega^2 + 1) \frac{\partial^2 J}{\partial z^2} +$$

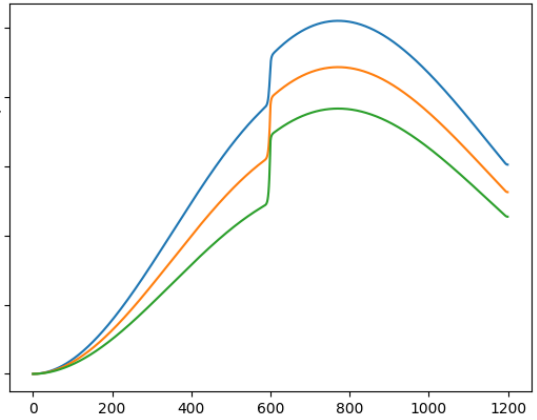
$$+ \left((\Omega^2 + 1) \frac{\partial \rho_0}{\partial r} + \rho_0 2\Omega \frac{\partial \Omega}{\partial z} \right) \frac{\partial J}{\partial z} = -2\pi r \left(\frac{\partial S_0}{\partial r} \frac{\partial T}{\partial z} - \frac{\partial T}{\partial r} \frac{\partial S_0}{\partial z} \right)$$

$$A_{zz} \frac{J_{i,k+1} - 2J_{ik} + J_{i,k-1}}{h^2}$$

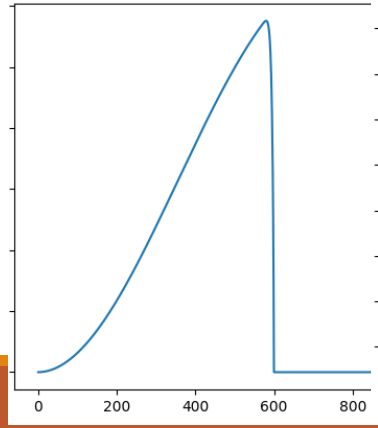


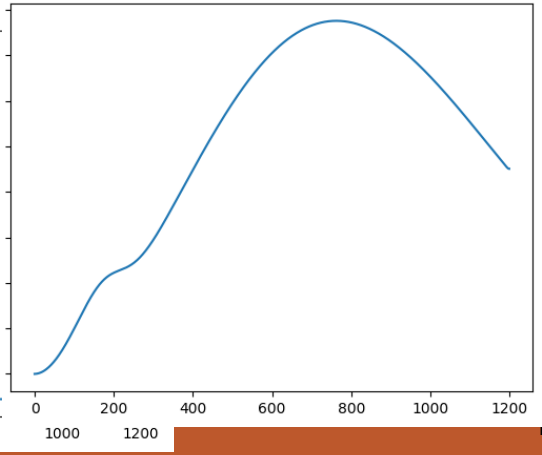
z_0





$$\rho_0 \frac{\partial^2 J}{\partial r^2} + \frac{\partial J}{\partial r} \left(\frac{\partial \rho_0}{\partial r} - \frac{\rho_0}{r} \right) + \rho_0 \frac{\partial^2 J}{\partial z^2} + \frac{\partial J}{\partial z} \frac{\partial \rho_0}{\partial z} = -2\pi r \left(\frac{\partial S_0}{\partial r} \frac{\partial T}{\partial z} - \frac{\partial T}{\partial r} \frac{\partial S_0}{\partial z} \right)$$





$$\rho_0 \frac{\partial^2 J}{\partial r^2} + \frac{\partial J}{\partial r}\left(\frac{\partial \rho_0}{\partial r} - \frac{\rho_0}{r}\right) + \left(\frac{\rho_1^2}{\rho_2} + \rho_2\right)\frac{\partial^2 J}{\partial z^2} + \frac{\partial J}{\partial z}\left(-\frac{\rho_1^2}{\rho_2^2}\frac{\partial \rho_2}{\partial z} + \frac{\partial \rho_2}{\partial z} + \frac{2\rho_1}{\rho_2}\frac{\partial \rho_1}{\partial z}\right) =$$
$$-2\pi r\left(-\frac{\rho_1}{\rho_2}\left(S_1 \frac{\partial^2 T}{\partial r\,\partial z} + \frac{\partial T}{\partial r}\frac{\partial S_1}{\partial z}\right) + (S_0 - S_2)\frac{\partial^2 T}{\partial r\,\partial z} - \frac{S_1}{\rho_2}\frac{\partial T}{\partial r}\frac{\partial \rho_1}{\partial z} + \frac{\rho_1 S_1}{\rho_2^2}\frac{\partial \rho_2}{\partial z}\frac{\partial T}{\partial r} + \left(\frac{\partial S_0}{\partial r}\frac{\partial T}{\partial z} - \frac{\partial T}{\partial r}\frac{\partial S_2}{\partial z}\right)\right)$$

$$\sigma = \frac{4}{3\sqrt{\pi}}\frac{q^2 n_e}{m}\int_0^\infty u^{3/2}\tau e^{-u}\begin{pmatrix} \frac{1}{1+\Omega^2} & -\frac{\Omega}{1+\Omega^2} & 0 \\ \frac{\Omega}{1+\Omega^2} & \frac{1}{1+\Omega^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{d}u$$

$$\rho = (\sigma)^{-1} \qquad u = \mathcal{E}/T$$

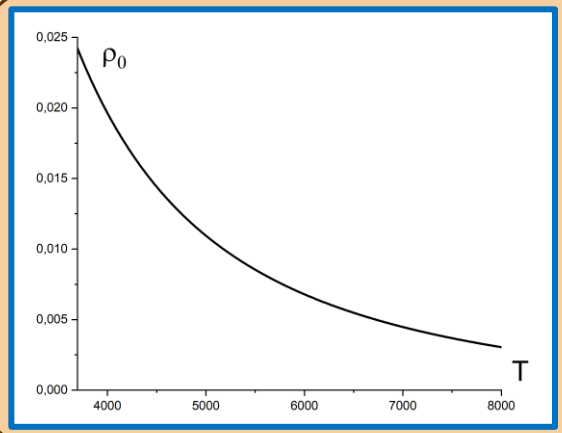
$$K = \sigma S = -\frac{4}{3\sqrt{\pi}}\frac{q n_e}{m}\int_0^\infty (\eta + u)u^{3/2}\tau e^{-u}\begin{pmatrix} \frac{1}{1+\Omega^2} & -\frac{\Omega}{1+\Omega^2} & 0 \\ \frac{\Omega}{1+\Omega^2} & \frac{1}{1+\Omega^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{d}u$$

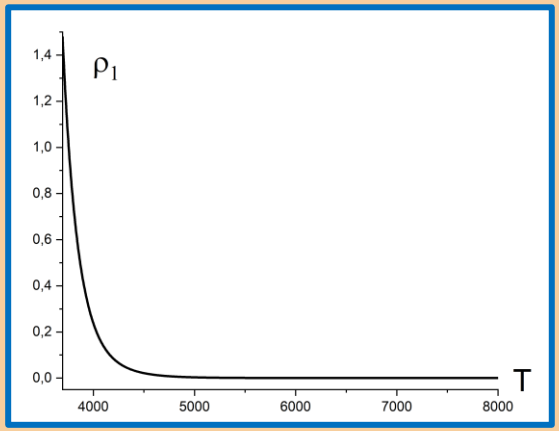
$$\vec{\Omega} = \frac{q\tau \vec{B}}{m}$$

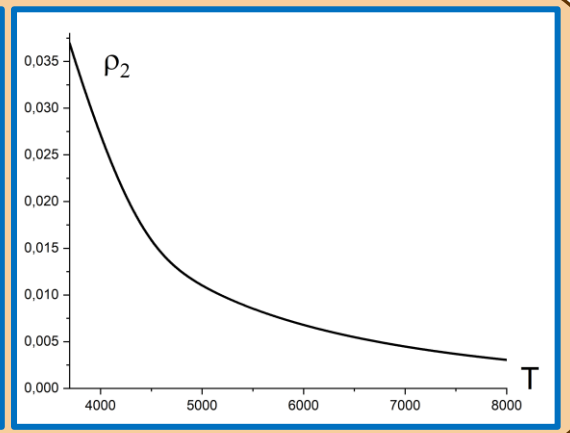
$$\frac{1}{\tau(u)} = \sqrt{\frac{2uT}{m}}\left(n_a 2\pi\sqrt{\frac{\alpha_e}{uT}} + n_i\frac{\pi}{2}\left(\frac{q^2}{uT}\right)^2 \log\left(1 + \left(\frac{2uT\sqrt{\frac{T}{2q^2 n_e}}}{q^2}\right)^2\right)\right)$$

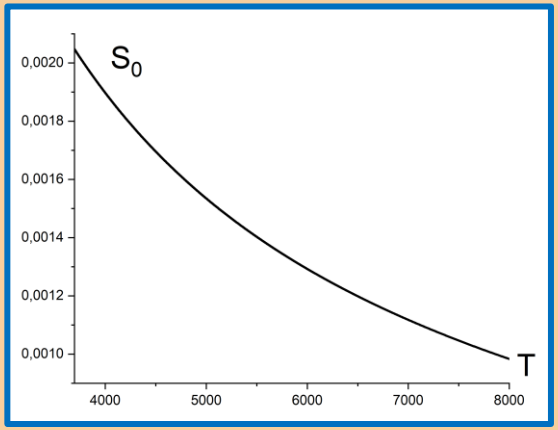
$$\begin{pmatrix} A_2 & -A_1 & 0 \\ A_1 & A_2 & 0 \\ 0 & 0 & A_0 \end{pmatrix}$$

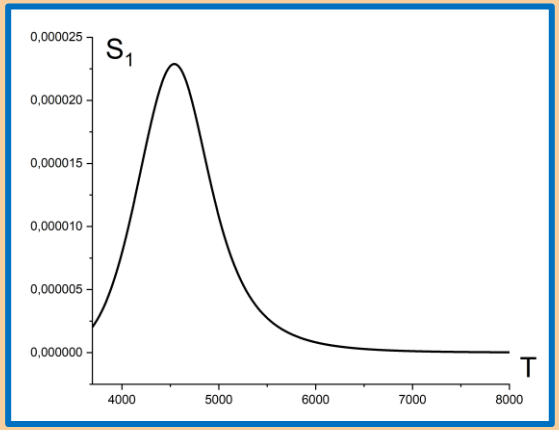
Термоэлектрические коэффициенты в газовой фазе

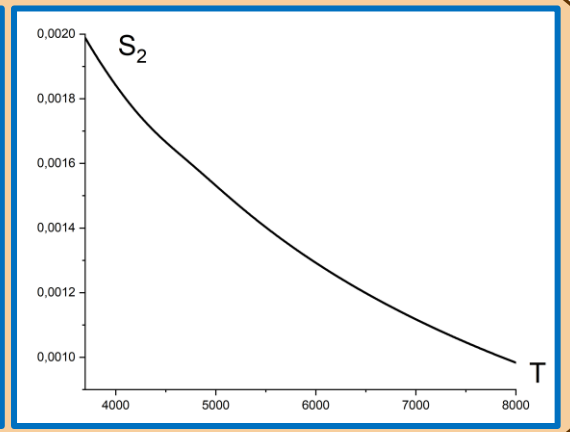


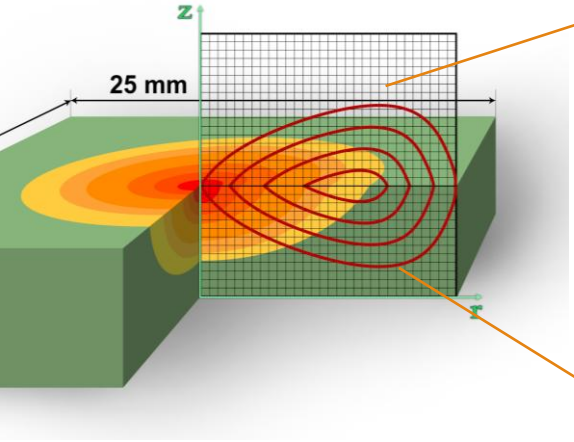


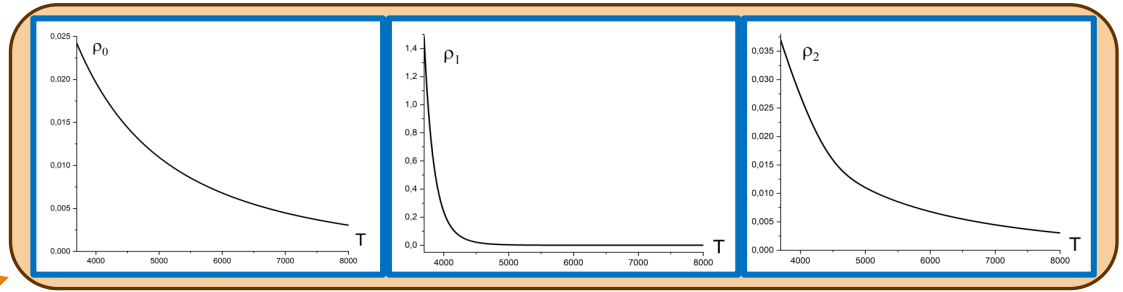


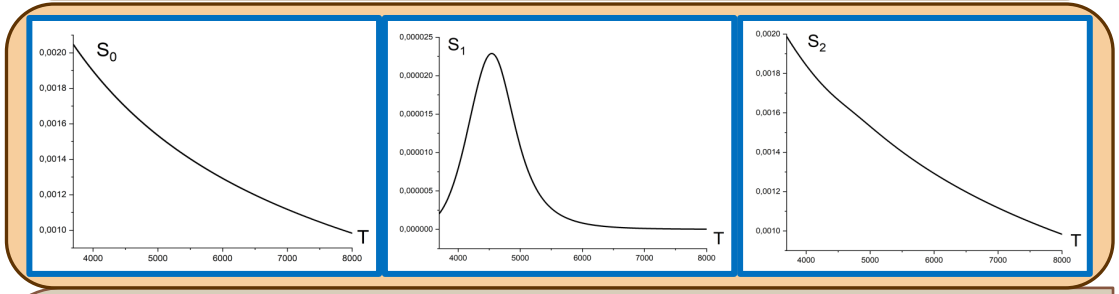




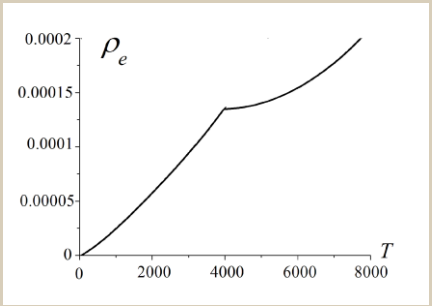


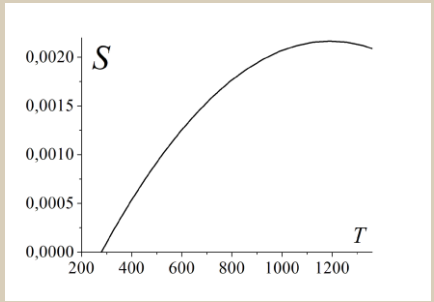






Экспериментальные данные





$$\frac{\partial^2 J}{\partial r^2}\rho_0 + \frac{\partial J}{\partial r}\left(\frac{\partial \rho_0}{\partial r} - \frac{\rho_0}{r}\right) + \frac{\partial^2 J}{\partial z^2}\left(\frac{\rho_1^2}{\rho_2} + \rho_2\right) + \frac{\partial J}{\partial z}\left(-\frac{\rho_1^2}{\rho_2^2}\frac{\partial \rho_2}{\partial z} + \frac{\partial \rho_2}{\partial z} + \frac{2\rho_1}{\rho_2}\frac{\partial \rho_1}{\partial z}\right)$$

$$= 2\pi r\left(\frac{\partial^2 T}{\partial r \partial z}\left(S_0 - S_2 - S_1\frac{\rho_1}{\rho_2}\right) + \frac{\partial T}{\partial r}\left(\frac{\rho_1 S_1}{\rho_2^2}\frac{\partial \rho_2}{\partial z} - \frac{S_1}{\rho_2}\frac{\partial \rho_1}{\partial z} - \frac{\rho_1}{\rho_2}\frac{\partial S_1}{\partial z} - \frac{\partial S_2}{\partial z}\right) + \frac{\partial S_0}{\partial r}\frac{\partial T}{\partial z}\right)$$

$$\frac{\partial T}{\partial r}\left(\frac{\rho_1 S_1}{\rho_2^2}[\rho_2]_{z=z_0} - \frac{S_1}{\rho_2}[\rho_1]_{z=z_0} - \frac{\rho_1}{\rho_2}[S_1]_{z=z_0} - [S_2]_{z=z_0}\right)\delta(z - z_0)$$

$$\frac{\partial J}{\partial z}\left(-\frac{\rho_1^2}{\rho_2^2}\frac{\partial \rho_2}{\partial z} + \frac{\partial \rho_2}{\partial z} + \frac{2\rho_1}{\rho_2}\frac{\partial \rho_1}{\partial z}\right)$$

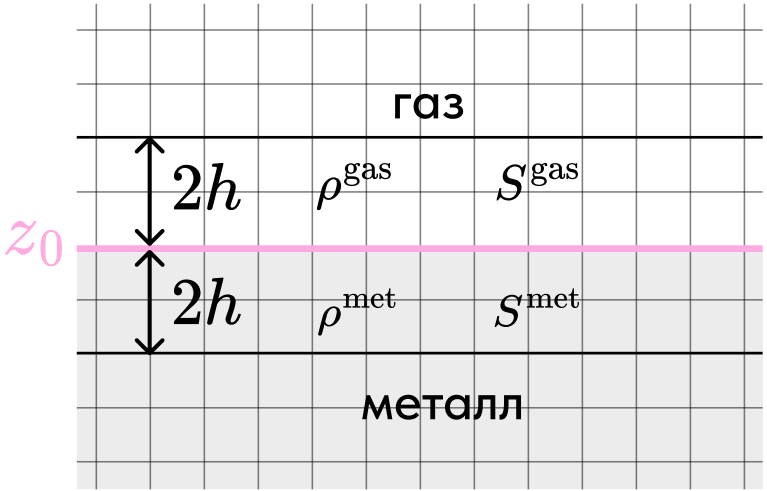
$$= \frac{\partial J}{\partial z}\left\{\left(1 - \frac{\rho_1^2}{\rho_2^2}\right)\frac{\partial \rho_2}{\partial z} + \frac{2\rho_1}{\rho_2}\frac{\partial \rho_1}{\partial z}\right\}_{z \le z_0 - \varepsilon,\ z \ge z_0 + \varepsilon} + \frac{\partial J}{\partial z}\left\{\left(1 - \frac{\rho_1^2}{\rho_2^2}\right)\frac{\partial P_2}{\partial z} + \frac{2\rho_1}{\rho_2}\frac{\partial P_1}{\partial z}\right\}_{z_0 - \varepsilon < z < z_0 + \varepsilon}$$

Регуляризация правой части

термоэмиссия

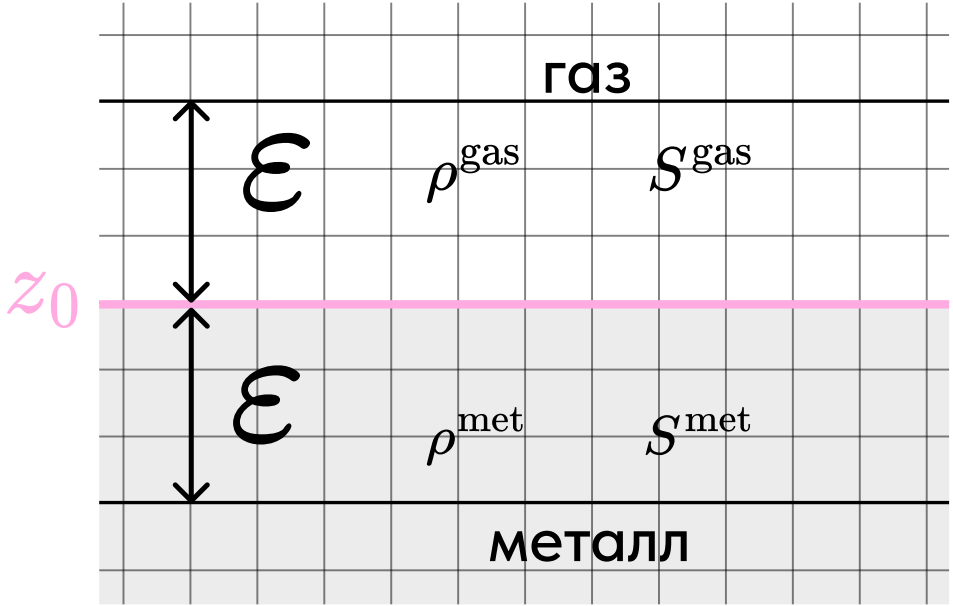
$$\frac{\partial T}{\partial r}\left(\frac{\rho_1 S_1}{\rho_2^2}[\rho_2]_{z=z_0} - \frac{S_1}{\rho_2}[\rho_1]_{z=z_0} - \frac{\rho_1}{\rho_2}[S_1]_{z=z_0} - [S_2]_{z=z_0}\right)\delta(z-z_0)$$

$$\delta_h(z-z_0) = \frac{1}{h} f\left(\frac{z-z_0}{h}\right)$$



$$f\left(\frac{z-z_0}{h}\right) = \begin{cases} \frac{1}{4}\left(1+\cos\left(\pi\frac{z-z_0}{2h}\right)\right), & \left|\frac{z-z_0}{h}\right| \le 2, \\ 0, & \left|\frac{z-z_0}{h}\right| > 2. \end{cases}$$

$$z \in [z_0 - 2h, z_0 + 2h]$$



Сглаживание разрывных коэффициентов

Регуляризация рассматривается как сглаживание разрыва в небольшой окрестности границы раздела сред

$$\frac{\partial J}{\partial z}\left(-\frac{\rho_1^2}{\rho_2^2}\frac{\partial \rho_2}{\partial z}+\frac{\partial \rho_2}{\partial z}+\frac{2\rho_1}{\rho_2}\frac{\partial \rho_1}{\partial z}\right)$$

$$=\frac{\partial J}{\partial z}\left\{\left(1-\frac{\rho_1^2}{\rho_2^2}\right)\frac{\partial \rho_2}{\partial z}+\frac{2\rho_1}{\rho_2}\frac{\partial \rho_1}{\partial z}\right\}_{z\le z_0-\varepsilon,\, z\ge z_0+\varepsilon}+\frac{\partial J}{\partial z}\left\{\left(1-\frac{\rho_1^2}{\rho_2^2}\right)\frac{\partial P_2}{\partial z}+\frac{2\rho_1}{\rho_2}\frac{\partial P_1}{\partial z}\right\}_{z_0-\varepsilon<z<z_0+\varepsilon}$$

Сигмоида

$$P_i = \frac{\rho_i^+ - \rho_i^-}{1 + e^{-\frac{z - z_0}{h}}} + \rho_i^-$$

Свёртка с B-сплайновым ядром

$$P_i = \sum_{z_0 - \varepsilon \le z_k \le z_0 + \varepsilon} K_h(z - z_0)\rho_i(z_k)$$

$$K_h(z - z_0) = \frac{1}{h} W\left(\frac{z - z_0}{h}\right)$$

$$W\left(\frac{z - z_0}{h}\right) = \begin{cases} \frac{2}{3} - \left|\frac{z - z_0}{h}\right|^2 + \frac{1}{2}\left|\frac{z - z_0}{h}\right|^3, & 0 \le \left|\frac{z - z_0}{h}\right| < 1, \\ \frac{1}{6}\left(2 - \left|\frac{z - z_0}{h}\right|\right)^3, & 1 \le \left|\frac{z - z_0}{h}\right| < 2, \\ 0, & \left|\frac{z - z_0}{h}\right| \ge 2. \end{cases}$$

Уравнение Гельмгольца

$$P_i - \varepsilon^2 \frac{\partial^2 P_i}{\partial z^2} = \rho_i$$

$$P_i\Big|_{z = z_0 - \varepsilon} = \rho_i^-$$

$$P_i\Big|_{z = z_0 + \varepsilon} = \rho_i^+$$

$$z \in [z_0 - \varepsilon, z_0 + \varepsilon]$$

Модельная задача

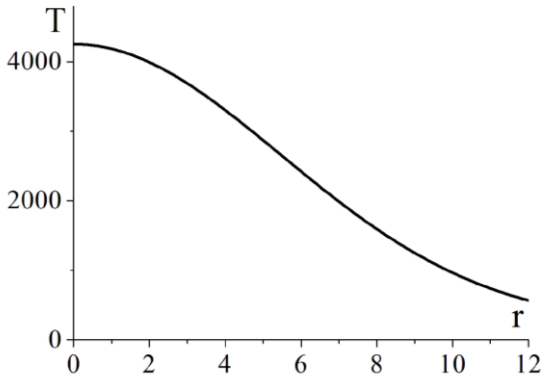
$$\rho_0^- = 1,5 \cdot 10^{-4}, \quad \rho_0^+ = 0,7 \cdot 10^{-3}, \quad S_0^- = 1,5 \cdot 10^{-5}, \quad S_0^+ = 10^{-3}.$$

Время свободного пробега постоянно относительно энергии электрона

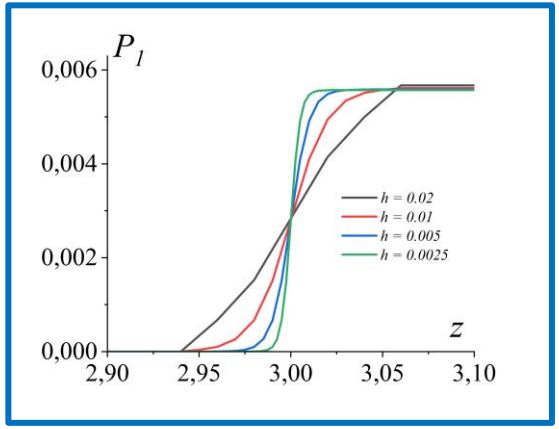
$$\Omega(T) = 10^{-4} \exp \frac{3(8000 - T)(9000 - T)}{2000T}.$$

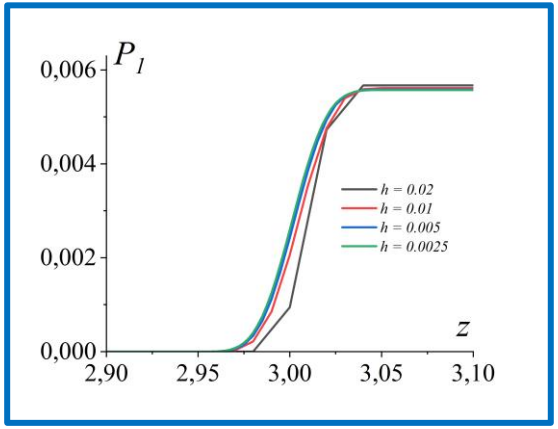
$$\rho_1 = \Omega\rho_0, \quad \rho_2 = \rho_0, \quad S_1 = 0, \quad S_2 = S_0.$$

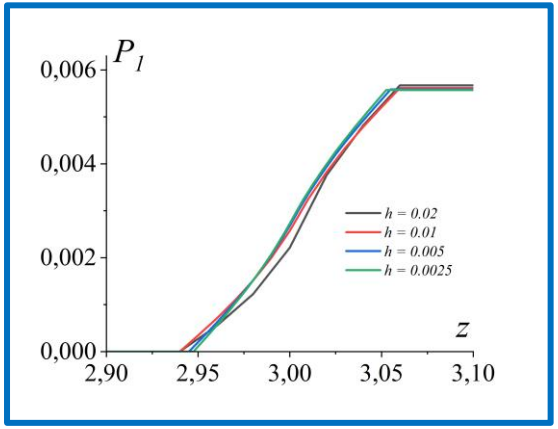
$$T(r, z_0) = T_0 + \frac{26}{2\pi} \cdot \exp\left(\frac{-r^2}{6000}\right).$$

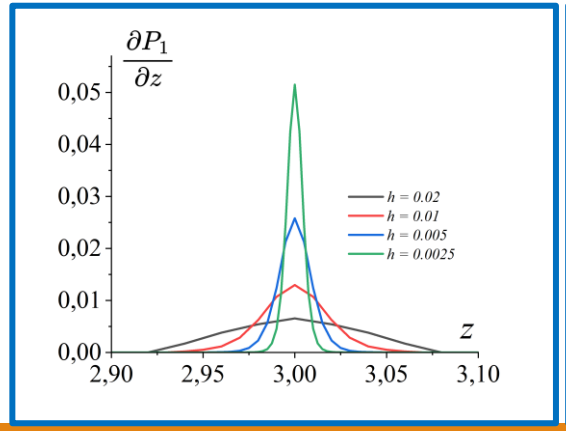


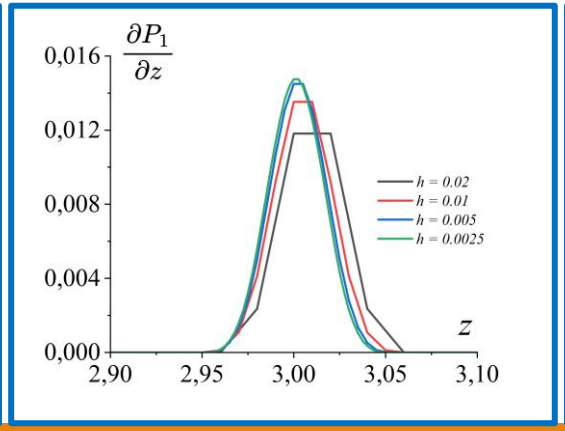
Сигмоида | **Свёртка** | **Ур-е Гельмгольца**

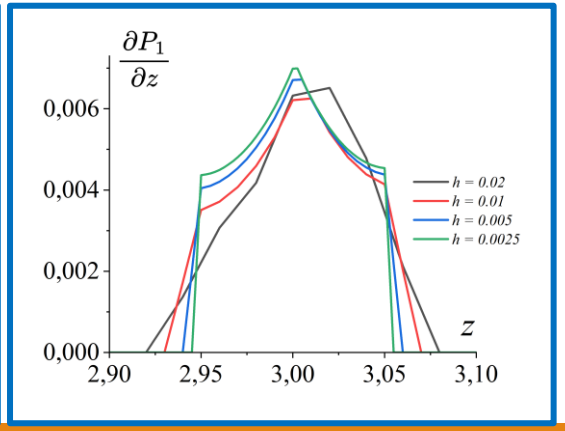








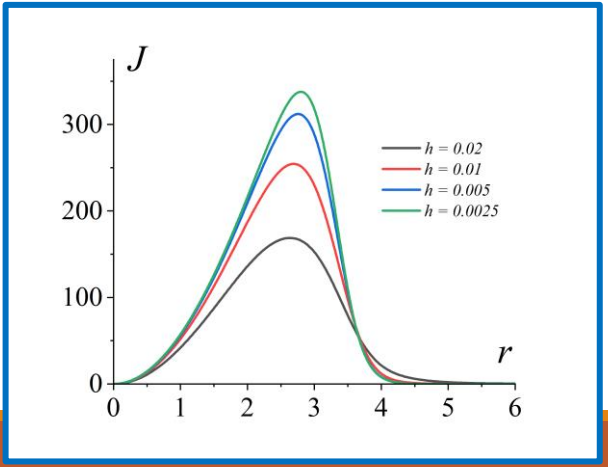




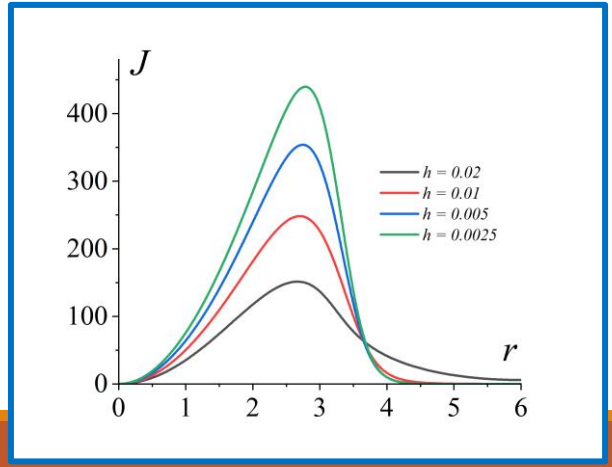
Профиль тока на поверхности образца

Шаг	Максимальное значение		
	Сигмоида	Свёртка	Ур-е Гельмгольца
0,02	168.756178	151.375325	174.587783
0,01	254.364612	248.317603	302.310445
0,005	312.051369	353.847974	413.467404
0,0025	337.680306	439.723248	486.814905

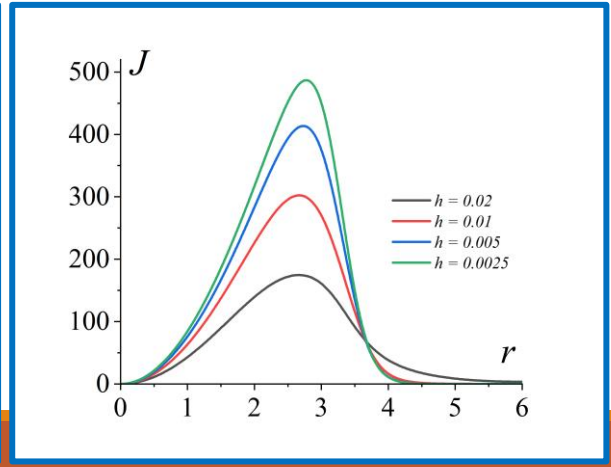
Сигмоида



Свёртка



Ур-е Гельмгольца



$$\frac{\partial^2 J}{\partial r^2}\rho_0 + \frac{\partial J}{\partial r}\left(\frac{\partial \rho_0}{\partial r} - \frac{\rho_0}{r}\right) + \frac{\partial^2 J}{\partial z^2}\left(\frac{\rho_1^2}{\rho_2} + \rho_2\right) + \frac{\partial J}{\partial z}\left(-\frac{\rho_1^2}{\rho_2^2}\frac{\partial \rho_2}{\partial z} + \frac{\partial \rho_2}{\partial z} + \frac{2\rho_1}{\rho_2}\frac{\partial \rho_1}{\partial z}\right)$$
$$= 2\pi r\left(\frac{\partial^2 T}{\partial r \partial z}\left(S_0 - S_2 - S_1\frac{\rho_1}{\rho_2}\right) + \frac{\partial T}{\partial r}\left(\frac{\rho_1 S_1}{\rho_2^2}\frac{\partial \rho_2}{\partial z} - \frac{1}{\rho_2}\frac{\partial(\rho_1 S_1)}{\partial z} - \frac{\partial S_2}{\partial z}\right) + \frac{\partial S_0}{\partial r}\frac{\partial T}{\partial z}\right)$$

Дивергентная форма уравнения

$$\frac{\partial}{\partial r}\left(\frac{\rho_0}{r}\frac{\partial J}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\left(\frac{\rho_1^2}{\rho_2} + \rho_2\right)\frac{\partial J}{\partial z}\right)$$
$$= 2\pi\left(\frac{\partial^2 T}{\partial r \partial z}\left(S_0 - S_2 - S_1\frac{\rho_1}{\rho_2}\right) + \frac{\partial T}{\partial r}\left(\frac{\rho_1 S_1}{\rho_2^2}\frac{\partial \rho_2}{\partial z} - \frac{1}{\rho_2}\frac{\partial(\rho_1 S_1)}{\partial z} - \frac{\partial S_2}{\partial z}\right) + \frac{\partial S_0}{\partial r}\frac{\partial T}{\partial z}\right).$$

$$\nabla \cdot (\beta(\mathbf{x})\nabla u(\mathbf{x})) = f(\mathbf{x}) + C(\mathbf{x})\delta(z - z_0)$$

$$[u]_{z=z_0} = u^+(\mathbf{x}) - u^-(\mathbf{x}) = 0,$$
$$[\beta u_n]_{z=z_0} = \beta^+(\mathbf{x})u_n^+(\mathbf{x}) - \beta^-(\mathbf{x})u_n^-(\mathbf{x}) = C(\mathbf{x})$$

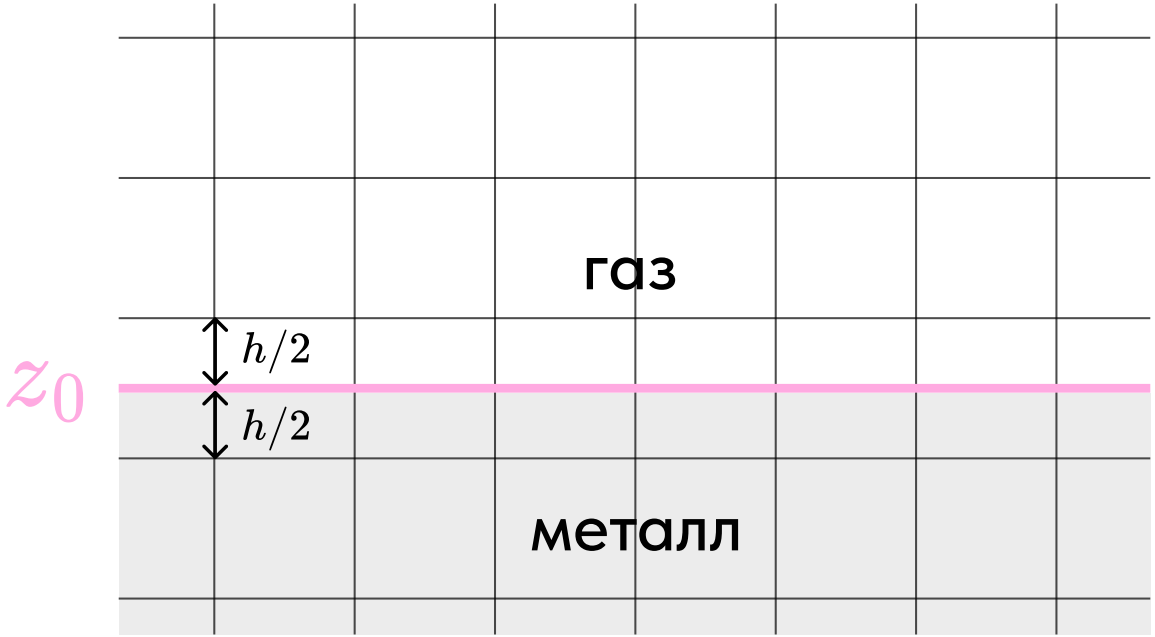
Внутренние граничные условия (условия сопряжения)

$$\mathbf{x} = (r, z) \qquad \beta(\mathbf{x}) = (\beta^r, \beta^z) = \left(\frac{\rho_0}{r}, \frac{1}{r}\left(\frac{\rho_1^2}{\rho_2} + \rho_2\right)\right)$$

$$f(\mathbf{x}) = 2\pi\left(\frac{\partial^2 T}{\partial r \partial z}\left(S_0 - S_2 - S_1\frac{\rho_1}{\rho_2}\right) + \frac{\partial S_0}{\partial r}\frac{\partial T}{\partial z} + \frac{\partial T}{\partial r}\left(\frac{\rho_1 S_1}{\rho_2^2}\frac{\partial \rho_2}{\partial z} - \frac{1}{\rho_2}\frac{\partial(\rho_1 S_1)}{\partial z} - \frac{\partial S_2}{\partial z}\right)_{z \neq z_0}\right)$$

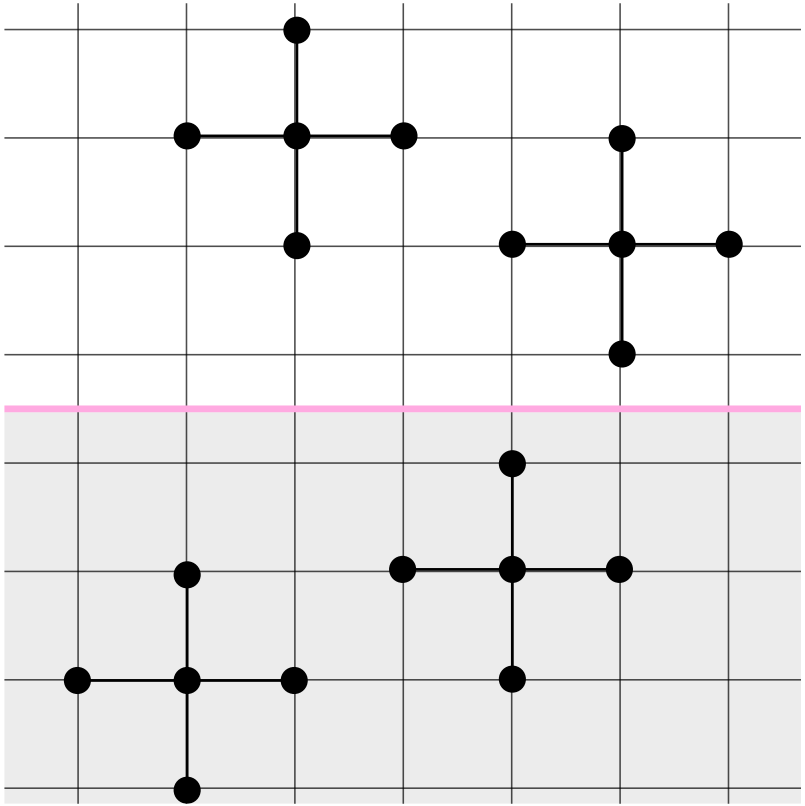
$$C(\mathbf{x})\delta(z - z_0) = 2\pi\frac{\partial T}{\partial r}\left(\frac{\rho_1 S_1}{\rho_2^2}[\rho_2]_{z=z_0} - \frac{S_1}{\rho_2}[\rho_1]_{z=z_0} - \frac{\rho_1}{\rho_2}[S_1]_{z=z_0} - [S_2]_{z=z_0}\right)\delta(z - z_0)$$

Метод погружённых интерфейсов
Immersed Interface Method (IIM)



Граница раздела сред не совпадает с расчётной сеткой

LeVeque R., Li Z. The immersed interface method for elliptic equations with discontinuous coefficients and singular sources // SIAM Journal on Numerical Analysis. 1994. V. 31, N. 4. P. 1019-1044

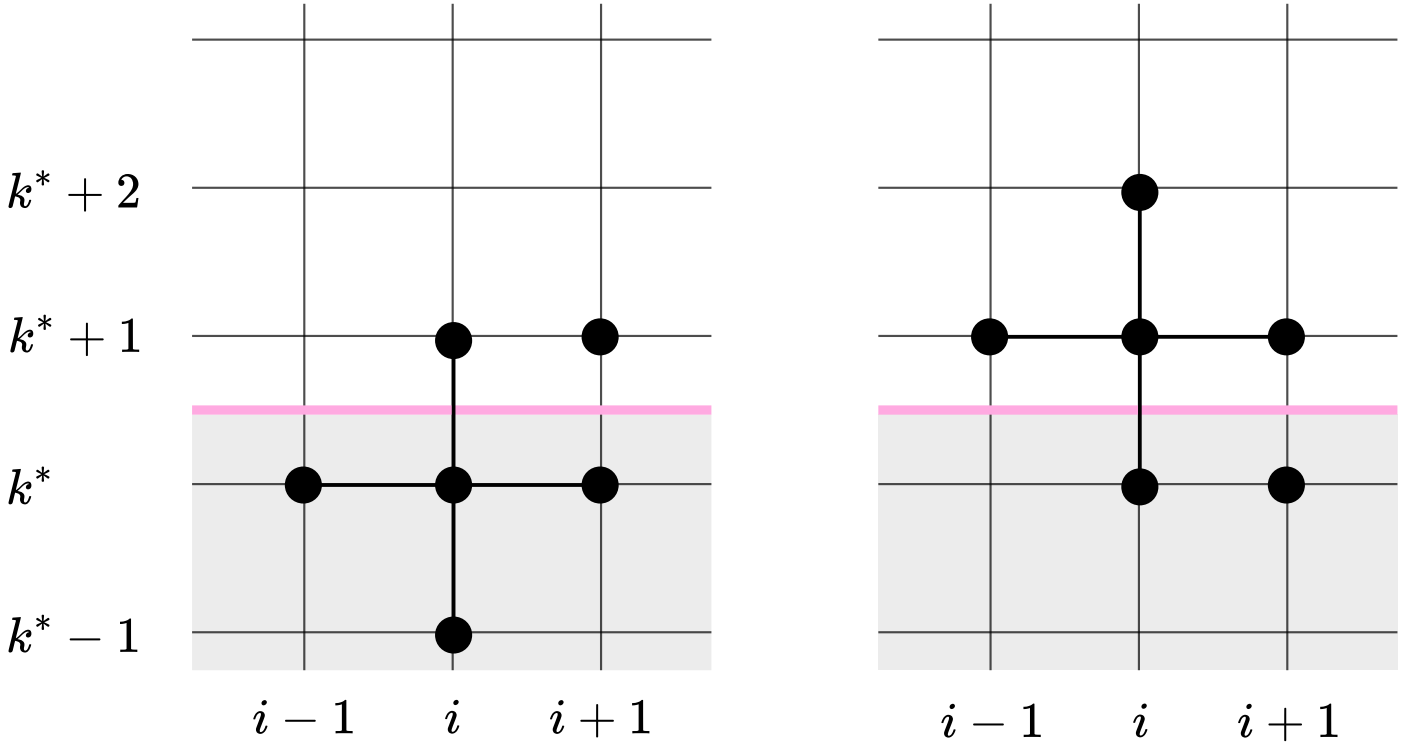


Регулярные узлы

$$\frac{1}{h}\left(\beta^r_{i+1/2,k}\frac{J_{i+1,k}-J_{ik}}{h}-\beta^r_{i-1/2,k}\frac{J_{ik}-J_{i-1,k}}{h}\right)$$
$$+\frac{1}{h}\left(\beta^z_{i,k+1/2}\frac{J_{i,k+1}-J_{ik}}{h}-\beta^z_{i,k-1/2}\frac{J_{ik}-J_{i,k-1}}{h}\right)=f_{ik}$$

LeVeque R., Li Z. The immersed interface method for elliptic equations with discontinuous coefficients and singular sources //
SIAM Journal on Numerical Analysis. 1994. V. 31, N. 4. P. 1019-1044

Нерегулярные узлы



LeVeque R., Li Z. The immersed interface method for elliptic equations with discontinuous coefficients and singular sources // SIAM Journal on Numerical Analysis. 1994. V. 31, N. 4. P. 1019-1044

6-точечная конечно-разностная схема

$$\gamma_1 J(r_{i-1}, z_k) + \gamma_2 J(r_i, z_k) + \gamma_3 J(r_{i+1}, z_k) + \gamma_4 J(r_i, z_{k-1}) + \gamma_5 J(r_i, z_{k+1}) + \gamma_6 J(r_{i+i_0}, z_{k+k_0}) = f_{ik} + C_{ik}$$

$$\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_6 \quad - \quad ? \qquad C_{ik} \quad - \quad ?$$

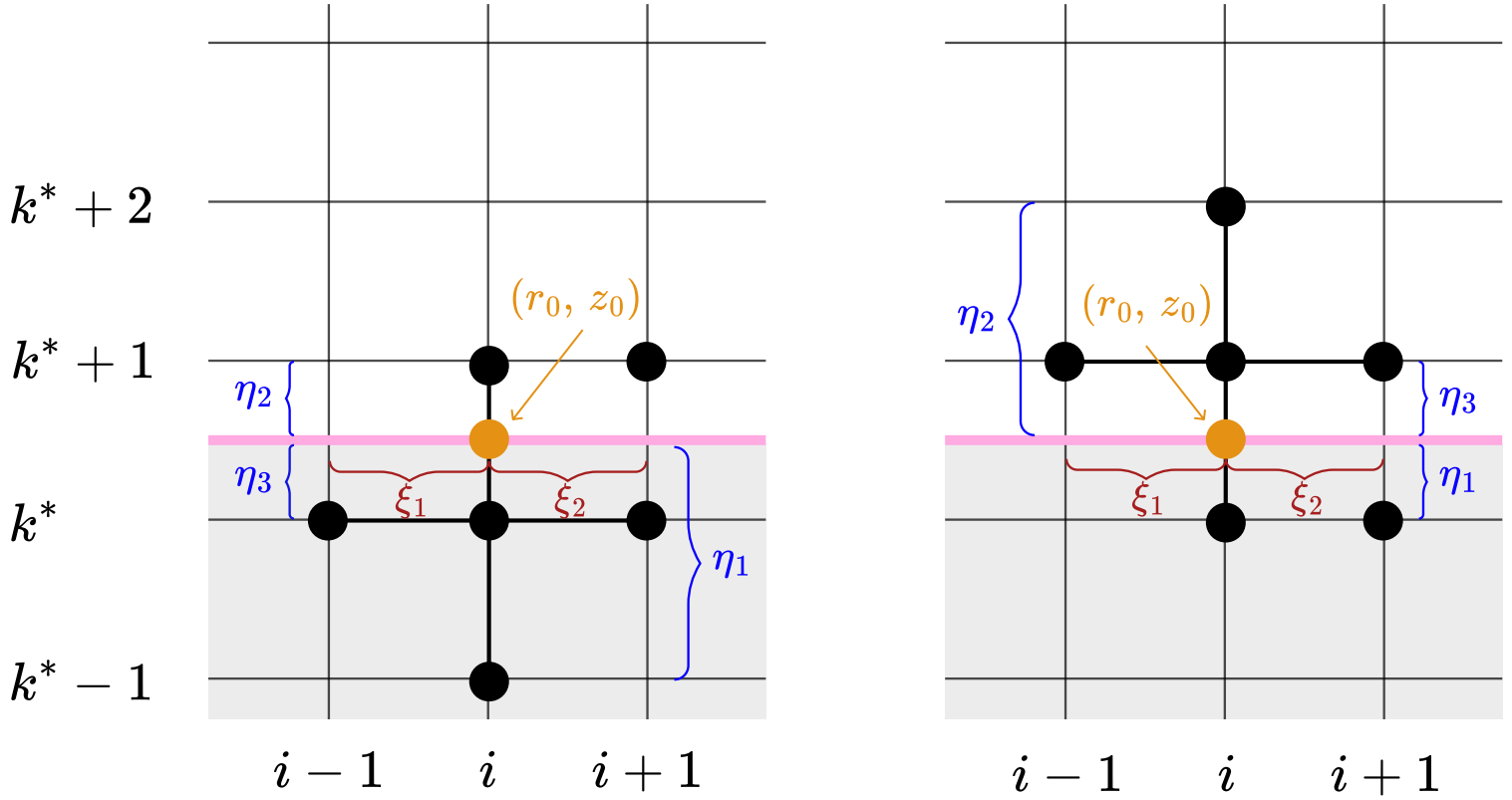
Локальная ошибка аппроксимации

$$R_{ik} = \gamma_1 J(r_{i-1}, z_k) + \gamma_2 J(r_i, z_k) + \gamma_3 J(r_{i+1}, z_k) + \gamma_4 J(r_i, z_{k-1}) + \gamma_5 J(r_i, z_{k+1}) + \gamma_6 J(r_{i+i_0}, z_{k+k_0}) - f_{ik} - C_{ik} = O(h)$$

Раскладываем функции в ряд Тейлора

LeVeque R., Li Z. The immersed interface method for elliptic equations with discontinuous coefficients and singular sources // SIAM Journal on Numerical Analysis. 1994. V. 31, N. 4. P. 1019-1044

Замена координат



LeVeque R., Li Z. The immersed interface method for elliptic equations with discontinuous coefficients and singular sources // SIAM Journal on Numerical Analysis. 1994. V. 31, N. 4. P. 1019-1044

Разложение в ряд Тейлора

$$J(r_i, z_{k^*}) = J^- + J_r^-(r_i - r_0) + J_z^-(z_{k^*} - z_0) + \frac{1}{2}J_{rr}^-(r_i - r_0)^2 + \frac{1}{2}J_{zz}^-(z_{k^*} - z_0)^2 + J_{rz}^-(r_i - r_0)(z_{k^*} - z_0) + O(h^3)$$

$$J(r_{i+1}, z_{k^*+1}) = J^+ + J_r^+(r_{i+1} - r_0) + J_z^+(z_{k^*+1} - z_0) + \frac{1}{2}J_{rr}^+(r_{i+1} - r_0)^2 + \frac{1}{2}J_{zz}^+(z_{k^*+1} - z_0)^2 + J_{rz}^+(r_{i+1} - r_0)(z_{k^*+1} - z_0) + O(h^3)$$

замена переменных

$$J(0, \eta_3) = J^- + J_\eta^-\eta_3 + \frac{1}{2}J_{\eta\eta}^-\eta_3^2 + O(h^3)$$

$$J(\xi_2, \eta_2) = J^+ + J_\xi^+\xi_2 + J_\eta^+\eta_2 + \frac{1}{2}J_{\xi\xi}^+\xi_2^2 + \frac{1}{2}J_{\eta\eta}^+\eta_2^2 + J_{\xi\eta}^+\xi_2\eta_2 + O(h^3)$$

В области ниже границы раздела выражаем все значения с “+” через значения с “–”, а в области выше – наоборот (с помощью исходного уравнения и условий сопряжения)

LeVeque R., Li Z. The immersed interface method for elliptic equations with discontinuous coefficients and singular sources // SIAM Journal on Numerical Analysis. 1994. V. 31, N. 4. P. 1019-1044

Область ниже границы раздела

$$
\begin{aligned}
R_{ik^*} = {} & (\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6)\cdot J^- \\
& + (\xi_1\gamma_1+\xi_2\gamma_3+K_1\gamma_5+(\xi_2+K_1)\gamma_6)\cdot J^-_{\xi} \\
& + (\eta_3(\gamma_1+\gamma_2+\gamma_3)+\eta_1\gamma_4+\eta_2K_2\gamma_5+\eta_2(K_2-\xi_2A_4)\gamma_6)\cdot J^-_{\eta} \\
& + \left(\frac{1}{2}\xi_1^2\gamma_1+\frac{1}{2}\xi_2^2\gamma_3+\frac{1}{2}K_3\gamma_5+\frac{1}{2}(\xi_2^2+K_3)\gamma_6\right)\cdot J^-_{\xi\xi} \\
& + \left(\frac{1}{2}\eta_3^2(\gamma_1+\gamma_2+\gamma_3)+\frac{1}{2}\eta_1^2\gamma_4+K_4\gamma_5+K_4\gamma_6\right)\cdot J^-_{\eta\eta} \\
& + (\xi_1\eta_3\gamma_1+\xi_2\eta_3\gamma_3+\xi_2\eta_2\rho\gamma_6)\cdot J^-_{\xi\eta} \\
& + \eta_2K\left(1-\frac{1}{2}\eta_2\frac{(\beta_2^+)_\eta}{\beta_2^+}\right)(\gamma_5+\gamma_6)+\frac{\xi_2\eta_2}{\beta_2^+}\left(C_\xi-K(\beta_2^+)_\xi\right)\gamma_6 \\
& - f^- - C_{ik^*}+O(h)
\end{aligned}
$$

$$
(\beta^r_r)^-J^-_{rr}+(\beta^z_z)^-J^-_{zz}+(\beta^r)^-J^-_r+(\beta^z)^-J^-_z=f^-
$$

$$
\rho=\frac{(\beta^z)^-}{(\beta^z)^+} \qquad K=\frac{C}{(\beta^z)^+}
$$

$$
A_1=\frac{(\beta^r)^+-(\beta^r)^-}{(\beta^z)^+} \qquad K_1=-\frac{1}{2}\eta_2^2A_3
$$

$$
A_2=\frac{\rho(\beta^z)^+_\eta-(\beta^z)^-_\eta}{(\beta^z)^+} \qquad K_2=\rho-\frac{1}{2}\eta_2A_2
$$

$$
A_3=\frac{(\beta^r)^+_\xi-(\beta^r)^-_\xi}{(\beta^z)^+} \qquad K_3=-\eta_2^2A_1
$$

$$
A_4=\frac{\rho(\beta^z)^+_\xi-(\beta^z)^-_\xi}{(\beta^z)^+} \qquad K_4=\frac{1}{2}\eta_2^2\rho
$$

Область выше границы раздела

$$
\begin{aligned}
R_{i,k^*+1} = {} & (\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6)\cdot J^+ \\
& + (\xi_1\gamma_1 + \xi_2\gamma_3 + K_1'\gamma_4 + (\xi_2+K_1')\gamma_6)\cdot J_\xi^+ \\
& + (\eta_3(\gamma_1+\gamma_2+\gamma_3) + \eta_1 K_2'\gamma_4 + \eta_2\gamma_5 + \eta_1(K_2'+\xi_2 A_4')\gamma_6)\cdot J_\eta^+ \\
& + \left(\frac{1}{2}\xi_1^2\gamma_1 + \frac{1}{2}\xi_2^2\gamma_3 + \frac{1}{2}K_3'\gamma_4 + \frac{1}{2}(K_3'+\xi_2^2)\gamma_6\right)\cdot J_{\xi\xi}^+ \\
& + \left(\frac{1}{2}\eta_3^2(\gamma_1+\gamma_2+\gamma_3) + K_4'\gamma_4 + \frac{1}{2}\eta_2^2\gamma_5 + K_4'\gamma_6\right)\cdot J_{\eta\eta}^+ \\
& + (\xi_1\eta_3\gamma_1 + \xi_2\eta_3\gamma_3 + \xi_2\eta_1\rho'\gamma_6)\cdot J_{\xi\eta}^+ \\
& + \eta_1 K'\left(\frac{1}{2}\eta_1\frac{(\beta_2^-)_\eta}{\beta_2^-} - 1\right)(\gamma_4+\gamma_6) + \frac{\xi_2\eta_1}{\beta_2^-}\left(K'(\beta_2^-)_\xi - C_\xi\right)\gamma_6 \\
& - f^+ - C_{i,k^*+1} + O(h)
\end{aligned}
$$

$$
\rho' = \frac{(\beta^z)^+}{(\beta^z)^-} \qquad K' = \frac{C}{(\beta^z)^-}
$$

$$
A_1' = \frac{(\beta^r)^+ - (\beta^r)^-}{(\beta^z)^-} \qquad K_1' = \frac{1}{2}\eta_1^2 A_3'
$$

$$
A_2' = \frac{(\beta^z)_\eta^+ - \rho'(\beta^z)_\eta^-}{(\beta^z)^-} \qquad K_2' = \rho' + \frac{1}{2}\eta_1 A_2'
$$

$$
A_3' = \frac{(\beta^r)_\xi^+ - (\beta^r)_\xi^-}{(\beta^z)^-} \qquad K_3' = \eta_1^2 A_1'
$$

$$
A_4' = \frac{(\beta^z)_\xi^+ - \rho'(\beta^z)_\xi^-}{(\beta^z)^-} \qquad K_4' = \frac{1}{2}\eta_1^2\rho'
$$

$$
(\beta_r^r)^+ J_{rr}^+ + (\beta_z^z)^+ J_{zz}^+ + (\beta^r)^+ J_r^+ + (\beta^z)^+ J_z^+ = f^+
$$

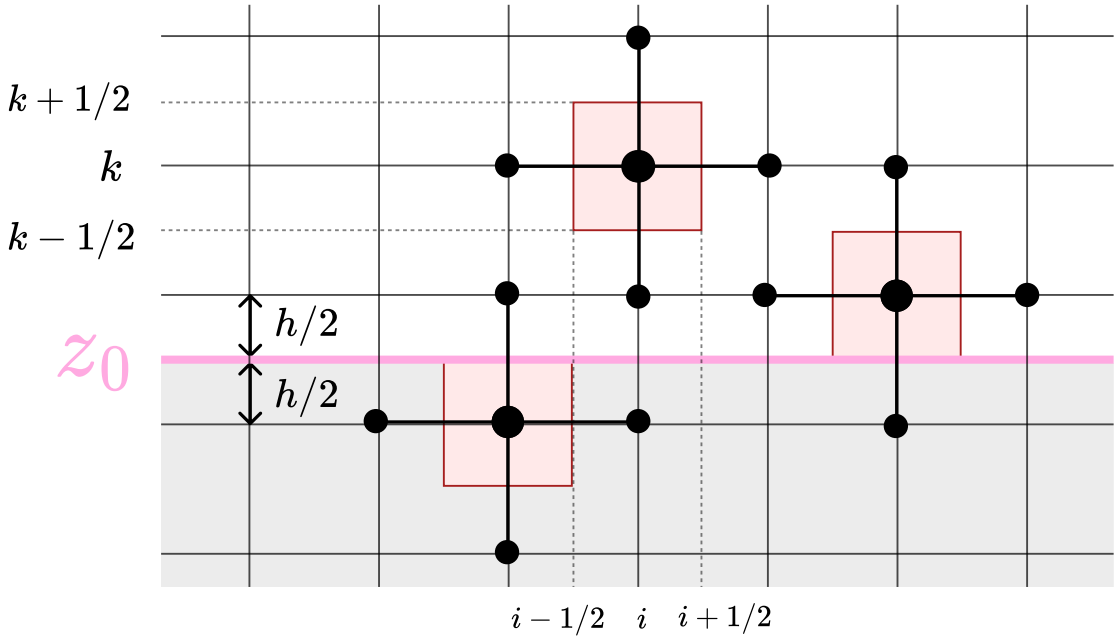
$$
\begin{aligned}
\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6 &= 0\\
\xi_1\gamma_1+\xi_2\gamma_3+K_1\gamma_5+(\xi_2+K_1)\gamma_6 &= 0\\
\eta_3(\gamma_1+\gamma_2+\gamma_3)+\eta_1\gamma_4+\eta_2K_2\gamma_5+\eta_2(K_2-\xi_2A_4)\gamma_6 &= (\beta_\xi^r)^-\\
\frac{1}{2}\xi_1^2\gamma_1+\frac{1}{2}\xi_2^2\gamma_3+\frac{1}{2}K_3\gamma_5+\frac{1}{2}(\xi_2^2+K_3)\gamma_6 &= (\beta_\eta^z)^-\\
\frac{1}{2}\eta_3^2(\gamma_1+\gamma_2+\gamma_3)+\frac{1}{2}\eta_1^2\gamma_4+K_4\gamma_5+K_4\gamma_6 &= (\beta^r)^-\\
\xi_1\eta_3\gamma_1+\xi_2\eta_3\gamma_3+\xi_2\eta_2\rho\gamma_6 &= (\beta^z)^-
\end{aligned}
$$

$$
\begin{aligned}
C_{ik^*} = \eta_2K\left(1-\frac{1}{2}\eta_2\frac{(\beta_2^+)_\eta}{\beta_2^+}\right)(\gamma_5+\gamma_6)\\
+\frac{\xi_2\eta_2}{\beta_2^+}\left(C_\xi-K(\beta_2^+)_\xi\right)\gamma_6
\end{aligned}
$$

$$
\begin{aligned}
\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6 &= 0\\
\xi_1\gamma_1+\xi_2\gamma_3+K_1'\gamma_4+(\xi_2+K_1')\gamma_6 &= 0\\
\eta_3(\gamma_1+\gamma_2+\gamma_3)+\eta_1K_2'\gamma_4+\eta_2\gamma_5+\eta_1(K_2'+\xi_2A_4')\gamma_6 &= (\beta_\xi^r)^+\\
\frac{1}{2}\xi_1^2\gamma_1+\frac{1}{2}\xi_2^2\gamma_3+\frac{1}{2}K_3'\gamma_4+\frac{1}{2}(K_3'+\xi_2^2)\gamma_6 &= (\beta_\eta^z)^+\\
\frac{1}{2}\eta_3^2(\gamma_1+\gamma_2+\gamma_3)+K_4'\gamma_4+\frac{1}{2}\eta_2^2\gamma_5+K_4'\gamma_6 &= (\beta^r)^+\\
\xi_1\eta_3\gamma_1+\xi_2\eta_3\gamma_3+\xi_2\eta_1\rho'\gamma_6 &= (\beta^z)^+
\end{aligned}
$$

$$
\begin{aligned}
C_{i,k^*+1} = \eta_1K'\left(\frac{1}{2}\eta_1\frac{(\beta_2^-)_\eta}{\beta_2^-}-1\right)(\gamma_4+\gamma_6)\\
+\frac{\xi_2\eta_1}{\beta_2^-}\left(K'(\beta_2^-)_\xi-C_\xi\right)\gamma_6
\end{aligned}
$$

Интегро-интерполяционный метод (Метод баланса)



Граница раздела сред не совпадает с расчётной сеткой

Для каждого узла сетки рассматривается ячейка, по которой уравнение интегрируется

$$G_{ik} = [r_{i-1/2}, r_{i+1/2}] \times [z_{k-1/2}, z_{k+1/2}],$$

$$W(r,z) = \left(W^r(r,z),\ W^z(r,z)\right) = \left(\beta^r(r,z)J_r(r,z),\ \beta^z(r,z)J_z(r,z)\right)$$

поток

$$\frac{\partial}{\partial r}\left(\beta^r \frac{\partial J}{\partial r}\right) + \frac{\partial}{\partial z}\left(\beta^z \frac{\partial J}{\partial z}\right) = f + C\delta(z - z_0)$$

$$\nabla \cdot W(r,z) = f + C\delta(z - z_0)$$

интегрируем по ячейке

$$\iint_{G_{ik}} \nabla \cdot W(r,z)\, dr\, dz = \iint_{G_{ik}} f\, dr\, dz + \iint_{G_{ik}} C\delta(z - z_0)\, dr\, dz$$

формула Остроградского-Гаусса для левой части

$$\int_{\partial G_{ik}} W(r,z) \cdot \mathbf{n}\, dr\, dz$$

$$\int_{\partial G_{ik}} W(r,z)\cdot\mathbf{n}\,dr\,dz$$

$$\int_{r_{i-1/2}}^{r_{i+1/2}} W^z(r,z_{k+1/2})\,dr - \int_{r_{i-1/2}}^{r_{i+1/2}} W^z(r,z_{k-1/2})\,dr + \int_{z_{k-1/2}}^{z_{k+1/2}} W^r(r_{i+1/2},z)\,dz - \int_{z_{k-1/2}}^{z_{k+1/2}} W^r(r_{i-1/2},z)\,dz$$

$$hW^z_{i,k+1/2} - hW^z_{i,k-1/2} + hW^r_{i+1/2,k} - hW^r_{i-1/2,k}$$

$$J_r(r,z) = \frac{W^r(r,z)}{\beta^r(r,z)}$$

$$J_{i,k} - J_{i-1,k} = \int_{r_{i-1}}^{r_i} \frac{W^r(r,z)}{\beta^r(r,z)}\, dr$$

$$W^r_{i-1/2,k} \approx \frac{J_{ik} - J_{i-1,k}}{h} \left[\frac{1}{h} \int_{r_{i-1}}^{r_i} \frac{dr}{\beta^r(r,z)} \right]^{-1}$$

$$J_{i+1,k} - J_{i,k} = \int_{r_i}^{r_{i+1}} \frac{W^r(r,z)}{\beta^r(r,z)}\, dr$$

$$W^r_{i+1/2,k} \approx \frac{J_{i+1,k} - J_{ik}}{h} \left[\frac{1}{h} \int_{r_i}^{r_{i+1}} \frac{dr}{\beta^r(r,z)} \right]^{-1}$$

$$W(r,z) = \Big(W^r(r,z),\ W^z(r,z) \Big)$$
$$= \Big(\beta^r(r,z) J_r(r,z),\ \beta^z(r,z) J_z(r,z) \Big)$$

$$J_z(r,z) = \frac{W^z(r,z)}{\beta^z(r,z)}$$

$$J_{i,k} - J_{i,k-1} = \int_{z_{k-1}}^{z_k} \frac{W^z(r,z)}{\beta^z(r,z)}\, dz$$

$$W^z_{i,k-1/2} \approx \frac{J_{ik} - J_{i,k-1}}{h} \left[\frac{1}{h} \int_{z_{k-1}}^{z_k} \frac{dz}{\beta^z(r,z)} \right]^{-1}$$

$$J_{i,k+1} - J_{i,k} = \int_{z_k}^{z_{k+1}} \frac{W^z(r,z)}{\beta^z(r,z)}\, dz$$

$$W^z_{i,k+1/2} \approx \frac{J_{i,k+1} - J_{ik}}{h} \left[\frac{1}{h} \int_{z_k}^{z_{k+1}} \frac{dz}{\beta^z(r,z)} \right]^{-1}$$

Регулярные узлы

$$\iint_{G_{ik}} f\,dr\,dz \approx h^2 f_{ik} \qquad \iint_{G_{ik}} C\delta(z-z_0)\,dr\,dz = 0$$

$$\frac{1}{h}\left(K_{i+1/2,k}\frac{J_{i+1,k}-J_{ik}}{h} - K_{i-1/2,k}\frac{J_{ik}-J_{i-1,k}}{h}\right) + \frac{1}{h}\left(K_{i,k+1/2}\frac{J_{i,k+1}-J_{ik}}{h} - K_{i,k-1/2}\frac{J_{ik}-J_{i,k-1}}{h}\right) = f_{ik}$$

$$K_{i-1/2,k} = \left[\frac{1}{h}\int_{r_{i-1}}^{r_i}\frac{dr}{\beta^r(r,z)}\right]^{-1} \qquad K_{i+1/2,k} = \left[\frac{1}{h}\int_{r_i}^{r_{i+1}}\frac{dr}{\beta^r(r,z)}\right]^{-1}$$

$$K_{i,k-1/2} = \left[\frac{1}{h}\int_{z_{k-1}}^{z_k}\frac{dz}{\beta^z(r,z)}\right]^{-1} \qquad K_{i,k+1/2} = \left[\frac{1}{h}\int_{z_k}^{z_{k+1}}\frac{dz}{\beta^z(r,z)}\right]^{-1}$$

Нерегулярные узлы

$$[W^z]_{z=z_0} = (W^z)^+ - (W^z)^- = C$$

$$(W^z)^+ = \frac{J_{i,k^*+1} - J_{i,k^*} + CK^-}{K^+ + K^-}$$

$$(W^z)^- = \frac{J_{i,k^*+1} - J_{i,k^*} - CK^+}{K^+ + K^-}$$

$$J_{i,k^*+1} - J_{i,0} = (W^z)^+ \int_{z_0}^{z_{k^*+1}} \frac{dz}{(\beta^z)^+}$$

$$J_z(r,z) = \frac{W^z(r,z)}{\beta^z(r,z)}$$

$$J_{i,k^*+1} - J_{i,k^*} = (W^z)^+ K^+ + (W^z)^- K^-$$

$$J_{i,0} - J_{i,k^*} = (W^z)^- \int_{z_{k^*}}^{z_0} \frac{dz}{(\beta^z)^-}$$

$$K^+ = \int_{z_0}^{z_{k^*+1}} \frac{dz}{(\beta^z)^+} \qquad K^- = \int_{z_{k^*}}^{z_0} \frac{dz}{(\beta^z)^-}$$

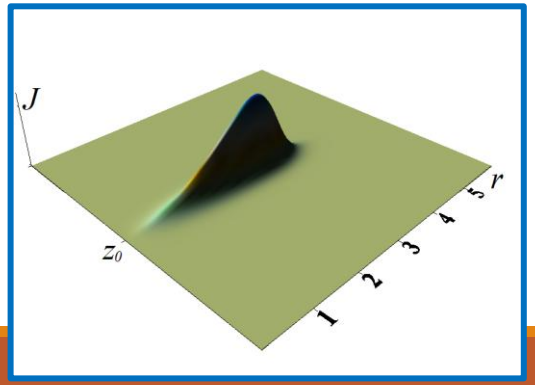
Нерегулярные узлы

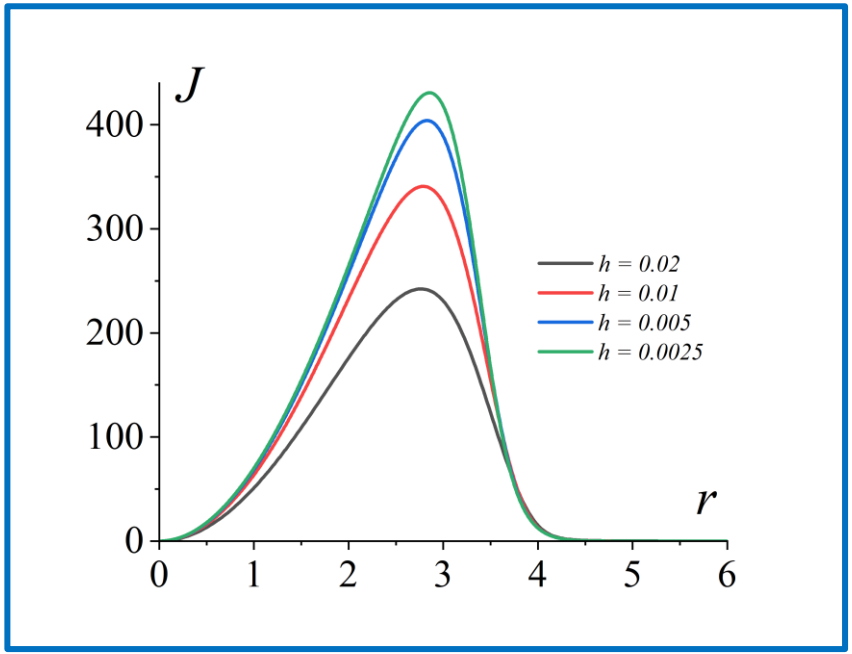
$$\frac{1}{h}\left(K_{i+1/2,k^*+1}\frac{J_{i+1,k^*+1}-J_{i,k^*+1}}{h}-K_{i-1/2,k^*+1}\frac{J_{i,k^*+1}-J_{i-1,k^*+1}}{h}\right)$$
$$+\frac{1}{h}\left(K_{i,k^*+3/2}\frac{J_{i,k^*+2}-J_{i,k^*+1}}{h}-\frac{1}{K^+ + K^-}\frac{J_{i,k^*+1}-J_{i,k^*}}{h}\right)=f_{i,k^*+1}+\frac{1}{h^2}\frac{CK^-}{K^+ + K^-}.$$

$$\frac{1}{h}\left(K_{i+1/2,k^*}\frac{J_{i+1,k^*}-J_{ik^*}}{h}-K_{i-1/2,k^*}\frac{J_{ik^*}-J_{i-1,k^*}}{h}\right)$$
$$+\frac{1}{h}\left(\frac{1}{K^+ + K^-}\frac{J_{i,k^*+1}-J_{ik^*}}{h}-K_{i,k^*-1/2}\frac{J_{ik^*}-J_{i,k^*-1}}{h}\right)=f_{ik^*}+\frac{1}{h^2}\frac{CK^+}{K^+ + K^-}$$

Распределение тока на поверхности при использовании интегро-интерполяционного метода (баланса) и IIM

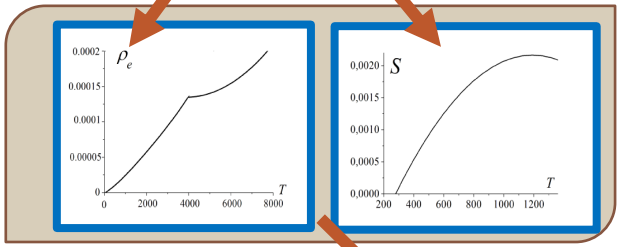
Шаг	Максимальное значение	
	IIM	Баланс
0,02	242.309488	242.312279
0,01	340.834155	340.834217
0,005	404.050473	404.050033
0,0025	430.759644	430.759177

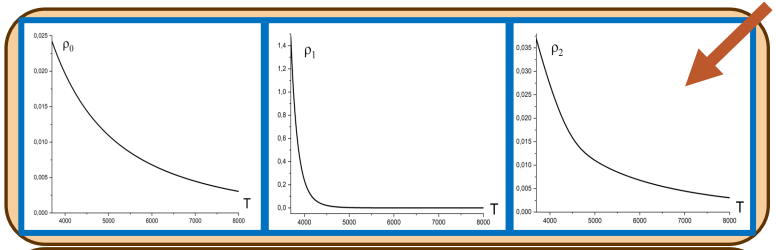


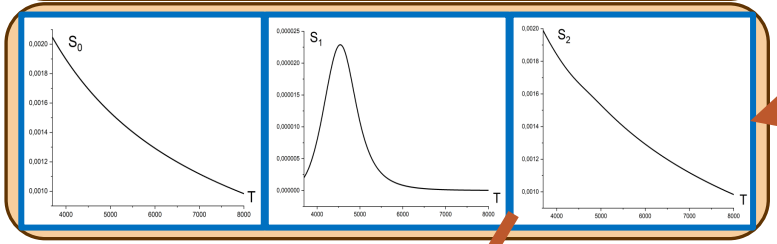


Шаг	Максимальное значение				
	Сигмоида	Свёртка	Ур-е Гельмгольца	IIM	Баланс
0,02	168.756178	151.375325	174.587783	242.309488	242.312279
0,01	254.364612	248.317603	302.310445	340.834155	340.834217
0,005	312.051369	353.847974	413.467404	404.050473	404.050033
0,0025	337.680306	439.723248	486.814905	430.759644	430.759177

$$\begin{cases} c(T)\rho^*(T)\dfrac{\partial T}{\partial t} = \dfrac{1}{r}\dfrac{\partial}{\partial r} r\lambda(T)\dfrac{\partial T}{\partial r} + \dfrac{\partial}{\partial z}\lambda(T)\dfrac{\partial T}{\partial z}, \\ \left(n, \nabla T\right)\Big|_\gamma = \dfrac{W(t,r) - N(t,r)}{\lambda(T)}, \quad \left(n, \nabla T\right) = 0, \quad T\big|_{t=0} = T_0. \end{cases}$$







$$\begin{cases} \dfrac{\partial \rho}{\partial t} + div\left(\rho\vec{v}\right) = 0, \\ \dfrac{\partial\left(\rho\vec{v}\right)}{\partial t} + div\left(\rho\vec{v}^2 + P\right) = 0, \\ \dfrac{\partial\left(\rho E\right)}{\partial t} + div\left(\rho E\vec{v} + P\vec{v}\right) = -\lambda(T)\, grad T, \\ T = (\gamma - 1)\dfrac{M}{R}\left(E - \dfrac{\vec{v}^2}{2}\right), \quad P = \dfrac{R}{M}\rho T, \\ v_z\big|_\gamma = \sqrt{\dfrac{5}{3}\dfrac{R}{M}}\sqrt{T\big|_\gamma}, \quad \rho\big|_\gamma = \dfrac{1}{2}\left(\dfrac{R}{M}\right)^{-1}\dfrac{1}{T\big|_\gamma}\exp\left(a1 - a2/T\big|_\gamma\right). \end{cases}$$

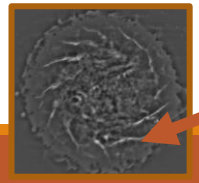
кластеризация

$$\rho_0\frac{\partial^2 J}{\partial r^2} + \frac{\partial J}{\partial r}\left(\frac{\partial \rho_0}{\partial r} - \frac{\rho_0}{r}\right) + \left(\frac{\rho_1^2}{\rho_2} + \rho_2\right)\frac{\partial^2 J}{\partial z^2} + \frac{\partial J}{\partial z}\left(-\frac{\rho_1^2}{\rho_2^2}\frac{\partial \rho_2}{\partial z} + \frac{\partial \rho_2}{\partial z} + \frac{2\rho_1}{\rho_2}\frac{\partial \rho_1}{\partial z}\right) =$$

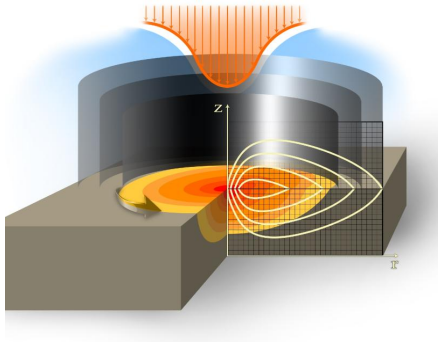
$$-2\pi r\left(-\frac{\rho_1}{\rho_2}\left(S_1\frac{\partial^2 T}{\partial r\,\partial z} + \frac{\partial T}{\partial r}\frac{\partial S_1}{\partial z}\right) + (S_0 - S_2)\frac{\partial^2 T}{\partial r\,\partial z} - \frac{S_1}{\rho_2}\frac{\partial T}{\partial r}\frac{\partial \rho_1}{\partial z} + \frac{\rho_1 S_1}{\rho_2^2}\frac{\partial \rho_2}{\partial z}\frac{\partial T}{\partial r} + \left(\frac{\partial S_0}{\partial r}\frac{\partial T}{\partial z} - \frac{\partial T}{\partial r}\frac{\partial S_2}{\partial z}\right)\right)$$

$$J\big|_{(0,z)} = 0, \quad \left.\frac{\partial J}{\partial z}\right|_{(r_{\max},z)} = 0, \quad \left.\frac{\partial J}{\partial r}\right|_{(r,z_{\max})} = 0, \quad J\big|_{(r,0)} = 0$$

$$\begin{cases} \nabla\times\vec{B} = \dfrac{4\pi}{c}\vec{j}, \\ \nabla\times\vec{E} = 0, \\ \nabla\cdot\vec{B} = 0, \\ \nabla\cdot\vec{j} = 0, \\ \vec{E} = \rho_e\vec{j}, \end{cases}$$



$$\begin{cases} \vec{j} = (j_r, 0, j_z) = \left(-\dfrac{\partial G}{\partial z}, 0, \dfrac{\partial G}{\partial r}\right), \\ g_\varphi = j_r B_z/\rho_1, \quad \rho_1 = 1.6\cdot 10^4\, kg/m^3, \quad B_z = 0.3 Tesla. \end{cases}$$



Дальнейшие планы:

1. Сокращение времени счёта уравнения эллиптического типа: использование метода бисопряжённых градиентов, многосеточные методы, использование пакетов программ
2. Другие методы регуляризации: Метод согласованных интерфейсов и границ, Matched Interface and Boundary (MIB)
3. Нахождение полуаналитического решения уравнения электростатики (Wang S., Lee A., Alexov E. Zhao S. A regularization approach for solving Poisson's equation with singular charge sources and diffuse interfaces. Applied Mathematics Letters. 2020. V. 102),
4. Сравнение с решением методом конечных элементов с использованием пакета COMSOL
5. Решение обратной задачи для определения эффектов кластеризации

Статьи:

1. *Lazareva G.G., Popov V.A., Yakunina E.N.* Mathematical Modeling of Thermoelectric Current Suppression outside the Melt Region of a Fusion Reactor Divertor Material. // Comput. Math. and Math. Phys., Vol. 66, No. 2, pp. 336-343, 2026.
2. *Lazareva G.G., Popov V.A., Yakunina E.N., Kasatov A.A., Cherepanov D., Kuzmin E.I.* Mathematical modeling of hydrogen diffusion in tungsten under model irradiation at the designed HIMP-2 facility // Lobachevskii Journal of Mathematics, 2026.

Регистрация программы для ЭВМ:

1. Свидетельство о государственной регистрации программы для ЭВМ № 2026689854 Российская Федерация. Программа расчёта динамики диффузии подвижного водорода в образце вольфрама в результате постоянного нагрева : № 2026688921 : заявл. 18.09.2026 : опубл. 30.09.2026 / Лазарева Г.Г., Якунина Е.Н.
2. Свидетельство о государственной регистрации программы для ЭВМ № 2026690365 Российская Федерация. Программа расчёта элементов тензора удельного сопротивления и термоЭДС в пристеночной плазме: № 2026688768 : заявл. 18.09.2026 : опубл. 05.10.2026 / Попов В.А., Якунина Е.Н.

Выступления на конференциях:

1. Международная научная конференция «Уфимская осенняя математическая школа – 2025» (УОМШ-2025). *Г.Г. Лазарева, В.А. Попов, Е.Н. Якунина*. Расчёт эффекта подавления термотоков вне зоны расплава материала дивертора термоядерного реактора.
2. Молодежная научная конференция «Новые горизонты прикладной математики» 2026. *Е.Н. Якунина, Г.Г. Лазарева, В.А. Попов*. Расчёт влияния ларморовской частоты на термотоки в материале дивертора термоядерного реактора.

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