Einstein Paradigm of Material Balance and Peaceman Well Block Problem For Time-Dependent Flows of Compressible Fluid

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Goal of the project

We consider sewing machinery between finite difference and analytical solutions defined at different scale: far away and near source of the perturbation of the of the flow. One of the essences of the approach is that coarse problem and boundary value problem in the proxy of the source model two different flows. We are proposing method to glue solution via total fluxes, which is predefined on coarse grid. It is important to mention that the coarse solution "does not see" boundary.
From industrial point of view our report can be considered as a mathematical "shirt" on famous Peaceman well-block radius formula for Darcy radial flow but can be applied in much more general scenario.

Peaceman Problem

Let $U = \bar{U} \setminus B_0$ be domain of flow generated by well-source $B_w = B(0, r_w) \subset B_{0,0} \subset \mathbb{R}^d$, $d =$ 1, 2.
Let $U_N = \cup B_{i,j}$ be discreet domain, of characteristic sizeΔ, and $u_N(s) = (u(i,j,s))\,, i,j = \cdots - 1, 0, +1 \cdots$ be the numerical solution as a matrix.
It is natural to assume that the block which doesn't contains source numerical value associate(close to) to average value of analytical solution.

Problem 1

How numerical value of $u(0,0,s)$ in the box $B_{0,0}$ associate to value of analytical solution of corresponding BVP on the well B_w .

To solve this problem we use sewing machinery based on the Material Balance (MB) equation.

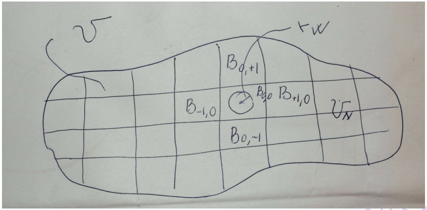


Figure 1: Domain Discretization and Numerical Solution

$B_{i,j},\ \ i,j=-1,\ 0,\ ,p_{i,j}$ characterise density in each box $B_{i,j}\ i=j\neq 0.$

$$\tau\cdot K_x^-\cdot(p_{-r_0,0}(s)-p_{-1,0}(s))=\tau\cdot q_x^-(s)+Q_x^-\,(p_{-r_0,0}(s+\tau)-p_{-r_0,0}(s))$$
$$\tau\cdot K_x^+\cdot(p_{r_0,0}-p_{1,0})=\tau\cdot q_x^+(s)+Q_x^+\,(p_{-r_0,0}(s+\tau)-p_{-r_0,0}(s))$$
$$\tau\cdot K_y^-\cdot(p_{0,-r_0}-p_{0,-1})=\tau\cdot q_y^-(s)+Q_y^-\,(p_{-r_0,0}(s+\tau)-p_{-r_0,0}(s))$$
$$\tau\cdot K_y^+\cdot(p_{0,r_0}-p_{0,1})=\tau\cdot q_y^+(s)+Q_y^+\,(p_{0,r_0}(s+\tau)-p_{0,r_0}(s))$$

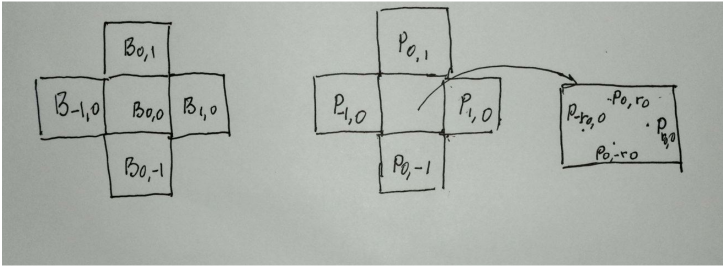


Figure 2: Material Balance Generic Einstein Model of Random Jumps

Geometrical Interpretation of Classical MB

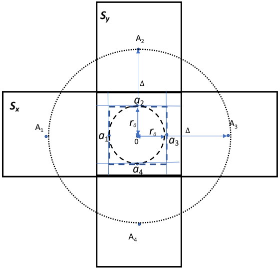


Figure 3: Einstein Mat balance equation on the 5 spots grid

Symmetric Flows in All

Denote:

$$q_x = q_x^- + q_x^+ \;\; q_y = q_y^- + q_y^+, \; Q_x = Q_x^- + Q_x^+ \;\; Q_y = q_y^- + Q_y^+, \quad (1.1)$$

and

$$q = q_x + q_y, Q = Q_x + Q_y, \quad (1.2)$$

Assume symmetry and anisotropy assumptions w.r.t. $+$ and $-$.

$$K_x^- = K_x^+ = K_x \; ; \; K_y^- = K_y^+ = K_y \quad (1.3)$$

$$q_x^- = q_x^+ = \frac{q_x}{2} \; q_y^- = q_y^+ = \frac{q_y}{2} \;\; cdots, \text{ and} \quad (1.4)$$

$$p_{-r_0,0} = p_{r_0,0} = p_{r_0}^x \; p_{-1,0} = p_{1,0} = p_1^x \quad (1.5)$$

$$p_{0,-r_0} = p_{0,r_0} = p_{r_0}^y \; p_{0,-1} = p_{0,1} = p_1^y$$

The goal of the project is to find R_0 which can be assigned to match calculated pressure in the block containing well to actual pressure at each point of the flow near well. Material Balance in general for transient flow has a form

$$4K \cdot (p_0(s) - p_1(s)) = -q + \varphi C_p \frac{V_0}{V} \cdot \frac{p_0(s+\tau) - p_0(s)}{\tau} \tag{1.6}$$

We consider three scenario

1. Steady State(SS)(This case was considered for Linear Darcy flow by Peaceman)
2. Pseudo State(PSS)
3. Boundary Dominated (BDD)

One dimensional and Two Dimensional Precursor for Material Balance

If one will assume that $(p^y_{r_0}(s) - p^y_1(s) = 0$, $\left(p^y_{r_0}(s+\tau) - p^y_{r_0}(s)\right)$, and$q_y(s) = 0$ then we will get a precursor for 1-D MB which in the case of symmetry in x - direction will take a form

$$\boxed{\tau \cdot 2 \cdot K_x \cdot \left(p^x_{r_0}(s) - p^x_1(s)\right) = \tau \cdot q_x(s) + Q_x \cdot 2\left(p^x_{r_0}(s+\tau) - p^x_{r_0}(s)\right).} \quad (1.7)$$

As a precursor for 2-D MB which in case of symmetry and anisotropy($p_{r_0} = p^x_{r_0} = p^y_{r_0}, \cdots$ and anisotropy: $K_x = K_y$ letting $q(s) = q_x(s) + q_y(s)$ and $Q(s) = Q_x(s) + Q_y(s)$ we will take a form

$$\boxed{\tau \cdot 4 \cdot K \cdot (p_{r_0}(s) - p_1(s)) = \tau \cdot q(s) + Q(s) \cdot 4\left(p_{r_0}(s+\tau) - p_{r_0}(s)\right).} \quad (1.8)$$

$$L \cdot (\Delta_x \cdot \Delta_y \cdot h) \cdot \left(p_{i,j}(t+\tau) - p_{i,j}(t)\right) = \tag{1.9}$$
$$\tau \cdot \left[Jh \left(\frac{\Delta_y}{\Delta_x}(p_{i-1,j}(t) - 2p_{i,j}(t) + p_{i+1,j}(t)) + \frac{\Delta_x}{\Delta_y}(p_{i,j-1}(t) - 2p_{i,j}(t) + p_{i,j+1}(t)) \right) + I\delta_{i,j} \right],$$
$$p = 0 \text{ on } \partial\Omega \times (-\infty, \infty).$$

Here $\delta_{i,j}$ is Kronecker symbol. Equation above is basic and can be applied in $1-D$ and $2-D$ cases, although has in both cases many similarities but it differ due to differences in the geometry of flow. Let thickness$h = 1$ of the reservoir is constant and

$$\boxed{I = q}. \tag{1.10}$$

then

1. Radial Material Balance Equation
   Under assumption of $2-D$ symmetry let

$$\boxed{\Delta = \Delta_x = \Delta_y}. \tag{1.11}$$

   Then equation (1.9) can be simplified as

$$L \cdot \Delta^2 \left(p_0(t+\tau) - p_{i,i}(t)\right) = \tau \left(4 \cdot J \cdot (p_1(t) - p_0(t)) + q\delta_{i,j}\right), \tag{1.12}$$
$$B(p) = 0 \text{ on } \partial\Omega \times (-\infty, \infty),\ B(\cdot) - \text{boundary operator}.$$

2. 1-D Material Balance
   Under assumption 1-D Symmetry let $\Delta_y = const$, and $\Delta = \Delta_x$ then MB takes form

$$L \cdot \Delta \cdot \Delta_y \left(p_0(t+\tau) - p_{i,i}(t)\right) = \tau \left(2 \cdot (J \cdot \Delta_y) \cdot \frac{(p_1(t) - p_0(t))}{\Delta} + q\delta_{i,j}\right), \tag{1.13}$$
$$B(p) = 0 \text{ on } \partial\Omega \times (-\infty, \infty),\ B(\cdot) - \text{boundary operator}.$$

2-D Steady State (SS), and Geometry of the Flow

Under symmetry and isotropic condition follows Basic Linear Balance Equation of the form

$$\boxed{4K \cdot (p_0 - p_1) = q} \tag{1.14}$$

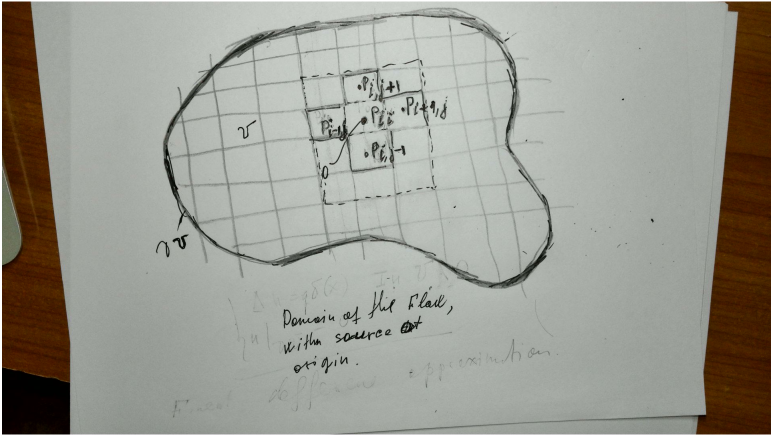
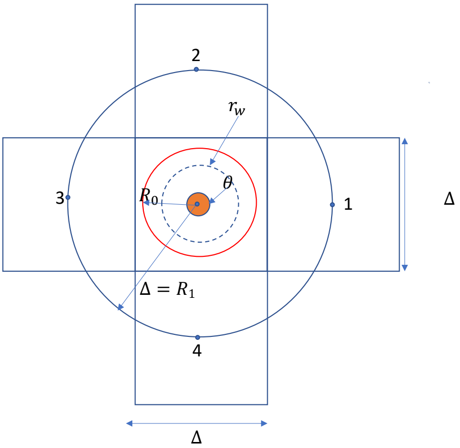


Figure 4: Domain of the Flow with source at 0

Back to Material Balance as Sewing Machinery

Under symmetry and isotropic condition follows Basic Linear Balance Equation of the form

$$4K \cdot (p_0 - p_1) = q \qquad (1.15)$$



Peaceman as Inverse Problem .

$$p_1 - p_0 = \alpha \frac{1}{4} q, \text{ here } \alpha = \frac{\mu}{kh}. \tag{1.16}$$

Analytical solution

$$p(r) = \alpha \frac{q}{2\pi} \ln \frac{r}{R_1} + p(R_1). \text{ here } \alpha = \frac{\mu}{kh}. \tag{1.17}$$

Peaceman Well-Posedness in can be stated as

Problem 2

Let value of p_1 and p_0 relate by material balance (1.16).*Let $\theta < R_0 < \Delta$. Find R_0 s.t.*

$$p(\theta) = \alpha \frac{q}{2\pi} \ln \frac{\theta}{R_0} + p_0, \tag{1.18}$$

and

$$p(\theta) = \alpha \frac{q}{2\pi} \ln \frac{\theta}{\Delta} + p_1. \tag{1.19}$$

Peaceman's Well Block Radius

Theorem 3

Assume that total rate of the production q and size of the grid Δ are given. Assume that single fully penetrated well located at the center of the numerical block $[-\frac{\Delta}{2}, \frac{\Delta}{2}]^2$. Let R_w is such that $\ln\frac{\Delta}{\theta} > \frac{\pi}{2}$ Then necessary and sufficient conditions that guarantee Peaceman well posedness is

$$\ln\frac{\Delta}{R_0} = \frac{\pi}{2}.$$

Interpretation in this section of the Peaceman paper is made directly without significant modification. But it is already clear that main aim is to sew numerical and analytical solutions formulated on the different scale to provide needed information for tuning procedure between numerical solution and observed Data throw relation

$$p(r) = \alpha\frac{q}{2\pi}\ln\frac{r}{R_0} + p_0$$

Another view on the problem

$$p_1 - p_0 = \frac{\alpha}{4} q \tag{1.20}$$

$$p_1 - p_w = \frac{\alpha}{2\pi} q \ln \frac{\Delta}{R_w} \tag{1.21}$$

$$p_0 - p_w = \frac{\alpha}{2\pi} q \ln \frac{R_0}{R_w} \tag{1.22}$$

Theorem 4

Assume that q and p_w solve equation for given p_1

$$q = \frac{2\pi k}{\mu} \frac{p_1 - p_w}{\ln \frac{\Delta}{R_w}} = 2\pi\alpha^{-1} \frac{p_1 - p_w}{\ln \frac{\Delta}{R_w}}. \tag{1.23}$$

Then, if R_0 satisfy equation

$$R_0 = \Delta \cdot e^{-\frac{\pi}{2}} \tag{1.24}$$

system (1.20)-(1.22) *has a solution for any mutually related p_1, p_0, and p_w*

Two Terms Forchheimer Peaceman

$$-\frac{\partial p}{\partial r} = \alpha_1 v_r + \beta v_r |v_r| \tag{1.25}$$

In (1.25) if $\beta = 0$ one can get classical Darcy equation. Due to $1 - D$ continuity equation radial velocity

$$v_r = -\frac{q}{2\pi r} \tag{1.26}$$

for any $r > 0$ if total rate (over well) q is fixed. From (1.25) and(1.26) follows

$$\frac{\partial p}{\partial r} = \alpha_1 \frac{q}{2\pi r} + \beta \frac{q}{4\pi^2 r^2} \tag{1.27}$$

$$p\big|_{r=R_2} - p\big|_{r=R_1} = f_2 - f_1 = \frac{\alpha_1 q}{2\pi} \ln \frac{R_2}{R_1} + \beta \frac{q}{4\pi^2} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \tag{1.28}$$

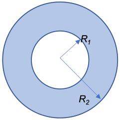


Figure 6: General Annual Domain U

Forchheimer Continue

Consider flow from $\partial B(0, R_2)$ to $\partial B(0, R_1)$ in the annular domain U(see Fig. 6):

$$\begin{cases} U = B(0, R_2) \setminus B(0, R_1), \text{ with fixed total rate } q = \int_S v(r) ds, , \\ \text{Given pressure on one of the boundaries } \partial B(0, R_i) : \\ p(r)\big|_{r=R_i} = f_i \text{ for } i = 1 \text{ or } 2. \end{cases} \tag{1.32}$$

From basic integration follows explicit formula for generic solution for two terms Forchheimer law:

$$p(r) = \frac{\alpha_1 q}{2\pi} \ln r - \beta \frac{q}{4\pi^2} \frac{1}{r} + constant \tag{1.33}$$

Then using boundary conditions in (1.32) one can get a generic formula for pressure depletion between two contours($\partial B(0, R_i)$) of the boundary of annular domain $U(0, R_1, R_2) = B(0, R_2) \setminus B(0, R_1)$.

We will hypothesise that on coarse greed material balance is still linear, whether near well correction is due to Forchheimer type of non-linearity. From Linear Material Balance Equation (1.20) and (1.28) with $\beta \neq 0$ follows the system of 3 equations:

$$p_1 - p_0 = \frac{\alpha}{4} q \tag{1.34}$$

$$p_1 - p_w = \frac{\alpha}{2\pi} q \ln \frac{\Delta}{R_w} + \beta \frac{q^2}{4\pi^2} \left(\frac{1}{R_w} - \frac{1}{\Delta} \right) \tag{1.35}$$

$$p_0 - p_w = \frac{\alpha}{2\pi} q \ln \frac{R_0}{R_w} + \beta \frac{q^2}{4\pi^2} \left(\frac{1}{R_w} - \frac{1}{R_0} \right) \tag{1.36}$$

Peaceman analogue of Well block radius for Non-linear Flow

Theorem 6

Assume that q and p_w solves quadratic equation

$$p_1 - p_w = \frac{\alpha}{2\pi} q \ln \frac{\Delta}{R_w} + \beta \frac{q^2}{4\pi^2} \left(\frac{1}{R_w} - \frac{1}{\Delta} \right) \tag{1.37}$$

Then, if R_0 satisfy equation

$$R_0 = \Delta \cdot e^{-\delta \frac{\pi}{2}} \tag{1.38}$$

system (1.34)-(1.36) *has a solution for any mutually related p_1, p_0, and p_w if δ satisfies equation*

$$\delta + \beta \frac{q}{\alpha\pi^2} \left(\frac{e^{\delta \frac{\pi}{2}}}{\Delta} - \frac{1}{\Delta} \right) = 1 \tag{1.39}$$

Engineering Findings for steady state (SS) case

Peaceman well block radius R_0^{ss} for Steady State (SS) MB, case $C_p = 0$ Let $p_{an}^{ss}(r)$ is pressure distribution of Steady state Problem in the reservoir then R_0^{ss} explicitly can be obtained on Δ geometric characteristic size of the grid, such that function p_{an}^{ss} obey Steady state material balance (SS-MB) namely

$$p_1 = p_{an}^{ss}(r)\Big|_{|r|=\Delta} \tag{1.40}$$

$$p_0 = p_{an}^{ss}(r)\Big|_{|x|=R_0^{ss}} \tag{1.41}$$

. Here p_1 and p_0, obtained from numerical simulation of the process on the grid of size Δ and R_0^{ss} to be found. It was proven the that

$$\boxed{R_0^{ss} = e^{-\frac{\pi}{2}} \cdot \Delta}. \tag{1.42}$$

R_0^{ss} does not depend on rate of the production and external radius of the reservoir R_e and well radius r_w.
This R_0^{ss} can be used

1. **to interpret numerically calculated P_0 for inverse problem**
2. **forecast value of the well pressure for direct problem**

We do not want in $R_0(t)$ for transient flows of slightly compressible fluid to be time dependent. For that we revisited model of the flow. It is engineering routine well classification consider flow which modeled in terms of pressure function $p(x, t)$ which in fact is subject to two IBVP

$$p_t^{(1,2)} = \Delta p^{(1,2)} \; ; \; \frac{\partial p^{1,2}}{\partial n}\bigg|_{\Gamma_e} = 0 \tag{1.43}$$

$$a) p^1|_{\Gamma_w} = p_w \; , \; b) \frac{\partial p^2}{\partial n}\bigg|_{\Gamma_w} = q \tag{1.44}$$

Productivity Index is functional

$$J_i(t) = \left(\ln \int_U p^i(x, t) dx - p_w^i\Big|_{\Gamma_w} \right)_t ; i = 1, 2 \tag{1.45}$$

It is not difficult to prove that

$$J_1, \text{is time ind. if } p^1 = e^{\lambda_0 t} \phi_0(x) \; \phi_0(x) - \text{first eigenfunction, and} \tag{1.46}$$

$$J_2, \text{is time ind if } p^2 = At + w_0(x), \Delta w_0(x) = A \tag{1.47}$$

$$-4K \cdot (p_0(s) - p_1(s)) + \frac{q}{h} = \Delta^2 \cdot 1 \cdot \frac{1}{\tau}(p_0(s+\tau) - p_0(s)), \quad (1.48)$$

Let the reservoir domain U with volume V, boundary $\partial U = \Gamma_e \cup \Gamma_w$ and thickness h.

Assumption 1

PSS constrain for slightly compressible fluid of compressiblity c_p.

1. $$(p_0(s+\tau) - p_0(s)) = q \cdot \frac{\tau}{1 \cdot V}, \quad (1.49)$$
2. $$p_0(s) - p_1(s) = constant\,(s)\,\text{independent.} \quad (1.50)$$

Under above constrain MB will take a form

$$4K \cdot (p_0(s) - p_1(s)) = \frac{q}{1} \cdot \left(1 - \frac{\Delta^2}{V}\right), \quad (1.51)$$

where q and is given constant and τ are time

PSS-Peaceman Formula

1. **Peaceman well block radius R_0^{ss} for Pseudo Steady State (PSS) MB** In order analytical PSS solution(**??**)to satisfy material balance (1.6) with constant production rate q it is sufficient

$$\boxed{-\pi + \frac{R_0^2}{r_e^2} = -2 \cdot \left(\ln \frac{\Delta}{R_0} \right)} \tag{1.52}$$

1. **Peaceman well block radius R_0^{ss} for Boundary Dominated Regime (BD) MB** In order analytical PSS solution to satisfy material balance with constant pressure value on the well and non-permeable external boundary it is sufficient
Peaceman well block radius R_0^{ss} for Boundary Dominated Regime (BD) MB In order analytical PSS solution to satisfy material balance with constant pressure value on the well and non-permeable external boundary it is sufficient

$$4 \cdot \left(\varphi_0(\lambda_0 \Delta) - \varphi_0(\lambda_0 R_0^{bd})\right) =$$

$$\left.\frac{\partial \varphi_0(\lambda_0 r)}{\partial r}\right|_{r=r_w} \cdot 2\pi r_w \cdot \Delta + \varphi_0(\lambda_0 \cdot R_0^{bd}) \cdot \frac{\phi c_p}{K} \cdot \frac{e^{-\lambda_0^2 \tau} - 1}{\tau}$$

These R_0^{BD} which deliver solution to transcendent equation depend on r_e, Δ and τ. Expectation is that for "small"$\tau,\ \tau << 1$, and big $r_e,\ r_e/r_w >> 1$ above formula can be well approximated by equation (1.53).
This equation is analogue of Peaceman formula for boundary dominated regime of the flow. By finding R_0 from this equation we provide correct value to calculate Peaceman well radius.

Peaceman Radius for BDD Flow for $\tau << 1$

1.

$$\boxed{4 \cdot (\varphi_0(\lambda_0 \Delta) - \varphi_0(\lambda_0 R_0^{bd})) = \frac{\partial \varphi_0(\lambda_0 r)}{\partial r}\Big|_{r=r_w} \cdot 2\pi r_w \cdot \Delta} \quad (1.53)$$

Here φ_0 is eigenfunction based on Bessel composition

$$\varphi_0(\lambda_0 r) = J_0(\lambda_0 r_w) \cdot N_0(\lambda_0 r) - J_0(\lambda_0 r) \cdot N_0(\lambda_0 r_w)\ , \quad (1.54)$$

and λ_0 root of the transcendent equation

$$\boxed{0 = J_0(\lambda_0 r_w) \cdot \frac{\partial N_0(\lambda_0 r)}{\partial r}\Big|_{r=r_e} - \frac{\partial J_0(\lambda_0 r)}{\partial r}\Big|_{r=r_e} \cdot N_0(\lambda_0 r_w)} \quad (1.55)$$

and $r_e,$ and r_w are exterior reservoir and well radius

2. **Very recent findings**

$$\boxed{R_0^{BD} \to R_0^{Peaceman} \ \text{as} \ \ r_e \to \infty} \quad (1.56)$$

Results above are generalised on Non-Linear flows

We divide all area of flow into $M \times M$ blocks. For all blocks(see fig . 4) , $0 \leq i \leq M$, $0 \leq j \leq M$. For the block of interest $Q_{i,j}$ For the Darcy flow can be reduced to the form:

$$\frac{kh\Delta y}{\mu\Delta x} \cdot (p_{i+1,j} - 2p_{i,j} + p_{i-1,j}) + \\ + \frac{kh\Delta x}{\mu\Delta y} \cdot (p_{i,j+1} - 2p_{i,j} + p_{i,j-1}) = q_{i,j} \tag{1.57}$$

In the above equation, $q_{i,j} = 0$ if $i \neq 0$ or $j \neq 0$($q_{i,j} = q \cdot \delta_{i,j}$-Kronecker symbol). Evidently size of the block in x and y direction are correspondingly Δx and Δy and are converging to 0 as $M \rightarrow \infty$. Let us denote $2M \times 2M$ matrix P_M

$$P_M = \left(p_{i,j}\right)_{((-M \leq i \leq M);(-M \leq j \leq M))} \tag{1.58}$$

Consider BVP in the bounded domain U containing source point 0(see fig . 4) :

$$-\nabla\left(\frac{kh}{\mu}\cdot\nabla p\right) = q^0\cdot\delta(x) \text{ in the domain } U \tag{1.59}$$

$$p(x) = 0 \text{ on the boundary } \partial U \tag{1.60}$$

Here $x = (x, y)$ and h thickness of the domain of flow. Elements of the matrix P_M represent values of of the solution of the discreet Poisson equation with RHS localised at center $(0, 0)$ stock/source. Let upgrade system by boundary condition.

Using classical machinery for Green function construction using Wieners approximation of the generalised solution I expect that following Conjecture can be proved.

Conjecture 7

Let $(x, y) \neq (0, 0)$ *fixed point of the domain* U. *This point belong to one of the element of the grid* U_M *which approximates domain* U. *Let* $p_M(x, y)$ *is solution of the system* $2M \times 2M$, *extended to be* $C^2(U) \cap C^0(\bar{U})$. *Then as* $M \to \infty$ *function* $p_M(x, y) \to G(x, y)$, *where*

$$G(x, y) = \frac{1}{2\pi} \ln \frac{1}{r} + g(x, y), \; r = \sqrt{x^2 + y^2}. \qquad (1.61)$$

is Green function.

Another more constructive approach on Green Function

The goal is is to compute the Green's function for the Laplace equation in the domain Ω for homogeneous Dirichlet boundary conditions

$$\begin{aligned} -\Delta U(x, x_0) &= \delta(x - x_0)\ ,\ x \in \Omega\ ,\ x_0 \in \Omega\ ; \\ U(x, x_0) &= 0 \text{ on } \Gamma = \partial\Omega \end{aligned} \tag{1.62}$$

Setting

$$U(x, x_0) = G(x - x_0) + \varphi(x, x_0). \tag{1.63}$$

where G is the fundamental solution

$$G(x - x_0) = -\frac{1}{2\pi} \ln |x - x_0|\ . \tag{1.64}$$

It follows that the corrector φ is the solution of

$$\begin{aligned} -\Delta\varphi(x, x_0) &= 0\ ,\ x \in \Omega\ ,\ x_0 \in \Omega\ ; \\ \varphi(x, x_0) &= -G(x - x_0) \text{ on } \Gamma = \partial\Omega \end{aligned} \tag{1.65}$$

This equation is homogeneous, and hence $\varphi(\cdot, x_0) \in C^\infty(\Omega)$. Moreover, since we assume that $x_0 \in \Omega$, the boundary data in (1.64) is a smooth function. However, the regularity of the solution φ in the closure $\bar{\Omega}$ depends on the smoothness of Γ. Here, we assume that

Green Function approximation

The idea of singularity correction is to solve the corrector equation with a standard numerical method, such as the usual five point finite difference approximation of the Laplacian, see, e.g., We denote by $\varphi_h(x, x_0)$ the approximation of $\varphi(x, x_0)$ at the grid points $x \in \Omega_h$, where h is the spacing. Then the following convergence result is well known

Theorem 9

If $\varphi \in C^4(\Omega)$ and R_h is the restriction to the grid Ω_h then

$$|\varphi_h - R_h\varphi|_\infty \leq \frac{h^2}{48}|\varphi|_{C^4(\Omega)}.$$

For a set A the oscillation of a function is defined as

$$osc_A f = \sup_{x\in A} f - \inf_{x\in A} f.$$

Property of the corrector and fundamental solution

Theorem 10

Suppose $R > 0$ is such that $B(x_0, R) \subset \Omega$, then there exist $C > 0$ depending on R only such that for any $r < R$

$$osc_{B(x_0,r)}\varphi(x, x_0) \leq C \cdot r \tag{1.67}$$

This follows from the smoothness of φ in the interior domain.

Theorem 11

For any $r_0 < r < R$ if

$$\ln \frac{r}{r_0} = \frac{\pi}{2} \tag{1.68}$$

then

$$4 \cdot \left(G(x, x_0)|_{x \in \partial B(x_0,r)} - G(y, x_0)|_{y \in \partial B(x_0,r_0)} \right) = 1 \tag{1.69}$$

From Theorems 1.67 and 11 follows

Theorem 12

Let r and r_0 the same as in Theorem 11, then

$$4 \cdot \left(U(x, x_0)|_{x \in \partial B(x_0, r)} - U(y, x_0)|_{y \in \partial B(x_0, r_0)} \right) = 1 + O(r) \qquad (1.70)$$

End for To Day

THANK YOU!!!