On the minimal residual methods for solving diffusion-convection SLAEs

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V P Il'in[1,2], D I Kozlov[2] and A V Petukhov[1]

[1] Institute of Computational Mathematics and Mathematical Geophysics SB RAS, pr. Akademika Lavrentjeva, 6, Novosibirsk, 630090, Russia
[2] Novosibirsk State University, Pirogova st., 1, Novosibirsk, 630090, Russia

E-mail: divinty5@gmail.com

Abstract. The objective of this research is to develop and to study iterative methods in the Krylov subspaces for solving systems of linear algebraic equations (SLAEs) with non-symmetric sparse matrices of high orders arising in the approximation of multi-dimensional boundary value problems on the unstructured grids. These methods are also relevant in many applications, including diffusion-convection equations. The considered algorithms are based on constructing A^TA — orthogonal direction vectors calculated using short recursions and providing global minimization of a residual at each iteration. Methods based on the Lanczos orthogonalization, A^T — preconditioned conjugate residuals algorithm, as well as the left Gauss transform for the original SLAEs are implemented. In addition, the efficiency of these iterative processes is investigated when solving algebraic preconditioned systems using an approximate factorization of the original matrix in the Eisenstat modification. The results of a set of computational experiments for various grids and values of convective coefficients are presented, which demonstrate a sufficiently high efficiency of the approaches under consideration.

1. Introduction

This paper deals with the topical problem of solving non-symmetric linear algebraic equations (SLAEs) with sparse matrices of high dimension ($10^8 - 10^{10}$ and higher), which arise when approximating various multi-dimensional boundary value problems of mathematical modeling in heat and mass transfer theory and other applications using finite difference methods, finite volumes, finite elements and the discontinuous Galerkin algorithms of various orders on unstructured grids [1]. The main tools here are iterative preconditioned processes in the Krylov subspaces, see [2] – [4] and numerous literatures cited therein.

The high rate of convergence of iterations in the cases under consideration is characteristic of the widespread methods of generalized minimal residuals (GMRES) and semi-conjugate residuals (SCR), which are studied in various versions and are accelerated by preconditioning using approximate matrix factorization, domain decomposition, graph transformations, and other approaches, whose survey is given in [5].

The main disadvantage of these algorithms is the need to store all direction vectors, which, for large dimensions and poor conditionality of algebraic systems, presents the high requirements for computational resources, both in the number of arithmetic operations and especially in the amount of memory, which presents significant problems even for modern supercomputers. The existing methods of using "restarts" and / or limiting the number of vectors to be memorized significantly expands the

possibilities of using such methods but leads to a significant decrease in the rate of convergence (and even degradation) of iterative processes.

Another class of iterative methods is based on biorthogonalization of the direction vectors using short (two-term or three-term) recurrence relations. This variety of approaches includes bi-conjugate directional algorithms (BiCG, BiCRStab, etc.) and versions of the IDR-processes (Induced Dimension Reduction), sometimes associated with the Sonneveld spaces, see [6] - [8]. In these algorithms, at each iteration, it is required to perform two matrix-vector multiplications, but their main drawback is the general absence of variational properties necessary to prove an optimal convergence, in the sense of minimizing any error functional at each iteration.

The same for the two families of algorithms under consideration is the property that, in the case of the symmetry of SLAEs, they turn into classical methods of conjugate directions (CG, CR – Conjugate Gradient, Conjugate Residual or methods of minimal errors associated with algebraic methods of moments described in [9] - [11]).

The purpose of this paper is to study methods for solving non-symmetric SLAEs that minimize the norm of a residual vector in the Krylov subspaces by A^TA-orthogonalization of the direction vectors using short recursions, whose implementation requires at each iteration one multiplication of the vector by A and by the transposed matrix A^T.

The approach in question is concretized using the three-term Lanczos orthogonalization procedure, the method of conjugate residuals $\mathrm{CRA^T}$ using the matrix A^T as a preconditioner, and also by applying to the original system

$$Au = f;\ u, f \in R^N, A \in R^{N,N} \tag{1}$$

the left Gauss transform:

$$\hat{A}u \equiv A^TAu = A^Tf;\ \check{A}v = AA^Tv = f, u = A^Tv, \tag{2}$$

with the subsequent solution of the obtained symmetric SLAEs by the method of conjugate residuals.

The efficiency of the algorithms under study is demonstrated on a representative series of algebraic systems formed using various finite difference approximations of the two-dimensional Dirichlet boundary value problem in the square for the diffusion-convection equation

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \mathrm{q}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = f(x,y), \tag{3}$$
$$x, y \in \Omega = [0,1]^2,\ \bar{\Omega} = \Omega \cup \Gamma, u|_\Gamma = g(x,y),$$

on the square grids

$$x_i = ih, \qquad y_j = jh; \qquad i,j = 0,1,\dots,M+1; \qquad h = (M+1)^{-1}. \tag{4}$$

2. Properties of the solved algebraic systems

We consider five-point systems of algebraic equations

$$(Au)_{i,j} = a_{i,j}u_{i-1,j} - b_{i,j}u_{i,j-1} - c_{i,j}u_{i+1,j} - d_{i,j}u_{i,j+1} + e_{i,j}u_{i,j} = f_{i,j}, \qquad i,j = 1,2,\dots,M,$$
$$a_{1,j} = b_{i,1} = c_{M,j} = d_{i,M} = 0, \tag{5}$$

approximating problem (3) on grid (4) in one of the three possible ways: using one-sided finite differences (OS), central differences (CD) and using an exponential scheme (EX) see [12]. For the case of a constant convective coefficient q, the values of nonzero off-diagonal matrix entries are presented in table 1, where $a_{i,j} = b_{i,j} = a,\ c_{i,j} = d_{i,j} = c$. Diagonal elements in all cases are determined by the relations

$$e_{i,j} = e \geq 2(a+c), \tag{6}$$

moreover, the strict inequality in (6) is satisfied only and only at the near-boundary grid nodes. Thus, if the values of a, c are non-negative, then the matrix A possesses the property of diagonal dominance and is monotone, i.e. $A^{-1} \geq 0$.

Table 1. Values of matrix entries for different grid approximations.

	OS	CD	EX
a	$1 + \mathrm{qh}$	$1 + \mathrm{qh}/2$	$\exp(-\mathrm{qh}/2)$
c	1	$1 - \mathrm{qh}/2$	$\exp(\mathrm{qh}/2)$

The matrix of the algebraic system (1), (5) can be represented as the sum

$$A = A^x + A^y, A^x = \text{block-diag}\left\{A_j^x\right\}, \tag{7}$$

where the matrices $\mathbf{A^x}, \mathbf{A^y}$ are tridiagonal, and if the coefficient q is constant, they are also commutative, i.e. $\mathbf{A^x A^y} = \mathbf{A^y A^x}$, and Toeplitz (non-zero elements on each diagonal are the same).

The diagonal blocks A_j^x in (7) are identical tridiagonal matrices of the following form:

$$A_j^x = 3\text{-diag}\{-a, e/2, -c\}. \tag{8}$$

Our further objective will be to obtain an estimate of the conditionality, which is necessary to study the rate of convergence of iterations:

$$cond(A^T A) = \max\{\lambda_k\} / \min\{\lambda_k\}, \tag{9}$$

where λ_k are the eigenvalues of the matrix $A^T A$ (in this case $\lambda_k = \nu_k^2$, where ν_k are the singular values of the matrix A).

The tridiagonal Toeplitz matrix A_j^x can be symmetrized using the similarity transformation

$$\bar{A}^x = D^{-1} A_j^x D = 3\text{-diag}\,(-\omega a, d = e/2, -\omega^{-1} c) = 3\text{-diag}\left(-\sqrt{ac}, d, -\sqrt{ac}\right), \;\; \omega = \sqrt{c/a}, \tag{10}$$

where $D = diag\{d_k = \omega^k\}$ is a diagonal matrix composed of a sequence of powers of the value ω. On the other hand, the symmetric matrix $\bar{A}^x$ for $a \neq c$ has the strong diagonal dominance

$$\delta = d - 2\sqrt{ac} > 0 \tag{11}$$

Because $d > a + c$ in this case. So, the condition number of this matrix can be estimated by means of Gershgorin circles as

$$cond(\bar{A}^x) \leq \frac{\Delta}{\delta}, \; \Delta = d + 2\sqrt{ac}. \tag{12}$$

Using the values of a, c from the table 1, we obtain

$$cond(\bar{A}^x) = O(q^{-2} h^{-2}), \; cond_{OS} > cond_{CD} > cond_{EX}. \tag{13}$$

Now, from the following inequality

$$cond(A^T A) = \|A^T A\| \|(A^T A)^{-1}\| \leq \|A^T\| \|A\| \|(A^T)^{-1}\| \|A^{-1}\| = cond(A^T) cond(A) \tag{14}$$

we have the condition number for the matrix product

$$cond((\bar{A}^x)^T \bar{A}^x) = O(q^{-4} h^{-4}).$$

And the similar results are valid for the matrix A^y also.

To support these theoretical estimates, table 2 shows the experimentally calculated values of condition numbers (13) for various values of the convective coefficient q and the number of the grid steps M along one coordinate (in each cell, from left to right, the values $cond((A^x)^T A^x)$ to approximate the types OS, CD and EX, respectively)

Table 2. Conditioning numbers of matrices for different types of approximation.

q\M	3			7			15		
0	5.82	5.82	5.82	25.27	25.27	25.27	103.08	103.08	103.08
1	5.79	5.68	5.55	25.05	24.32	20.97	102.04	98.9	56.94
2	5.73	5.31	4.89	24.51	22.12	14.04	99.35	89.31	24.57
4	5.57	4.5	3.54	23.08	17.52	6.37	91.64	69.61	7.78
8	5.31	3.67	2.15	20.48	12.06	2.41	76.86	46.42	2.44

3. Algorithms of minimal residuals for non-symmetric SLAEs

To solve the algebraic system (1), consider an iterative process of the form (see [3])

$$\begin{gathered} p^0 = r^0 = f - Au^0, \qquad n = 0, 1, \dots : \\ u^{n+1} = u^n + \alpha_n p^n, \qquad r^{n+1} = r^n - \alpha_n A p^n, \end{gathered} \tag{15}$$

where u^0 is an arbitrary initial approximation, r^n is the residual vector, and p^n are the direction vectors, with respect to which we assume that the following orthogonalization conditions are satisfied:

$$\left(Ap^k, Ap^n\right) = \left(p^k, A^T A p^n\right) = \rho_n \delta_{k,n}, \qquad \rho_n = (Ap^n, Ap^n). \tag{16}$$

It is easy to check that when determining the iterative parameters in (14) by formula

$$\alpha_k = \left(r^0, Ap^k\right)/\rho_n, \qquad k = 0,1, \dots, n, \tag{17}$$

the norms $\|r^{n+1}\|_2^2 = (r^{n+1}, r^{n+1})$ are minimized in the Krylov subspaces

$$K_{n+1}(p^0, A) = Span p^0, Ap^0, \dots, A^n p^0. \tag{18}$$

If conditions (16) are provided by the direct Gram-Schmidt orthogonalization

$$p^{n+1} = r^{n+1} - \sum_{k=0}^{n} \beta_{n,k} p^n, \quad \beta_{n,k} = \left(Ar^{n+1}, Ap^k\right)/\rho_n, \tag{19}$$

then we arrive at the SCR method of semi-conjugate residuals, which is equivalent in convergence rate to GMRES [2]. If the matrix $A = A^T$ is symmetric, formulas (17), (19) take the following simpler form:

$$p^{n+1} = r^{n+1} + \beta_n p^n, \qquad \beta_n = \sigma_{n+1}/\sigma_n, \qquad \sigma_n = (Ar^n, r^n), \qquad \alpha_n = \sigma_n/\rho_n, \tag{20}$$

which together with (15) defines the classical method of the conjugate residuals (CR).

As a rule, to speed up iterative methods in Krylov subspaces, the original system (1) is preliminarily modified by the left, the right, or two-sided preconditioning. For example, if the preconditioning matrix is represented as a product $B = \hat{B}\check{B}$ with non-degenerate easily invertible factors, then instead of (1) one can solve the SLAE

$$\tilde{A}\tilde{u} \equiv \check{B}^{-1} A \hat{B}^{-1} \tilde{u} = \tilde{f} \equiv \check{B}^{-1} f, \tilde{u} = \hat{B} u. \tag{21}$$

Let the original matrix be represented as a sum $A = D + L + U$, where D, L and U are diagonal (or block-diagonal), lower and upper triangular matrices, respectively. Following the USSOR non-symmetric upper relaxation method, or incomplete factorization, the preconditioning matrices are defined as

$$\check{B}^{-1} = \check{G}(G + L)^{-1}, \qquad \hat{B}^{-1} = (G + U)^{-1}\hat{G}, \qquad G = \check{G}\hat{G}. \tag{22}$$

Then the matrix of the preconditioned SLAE is written in the form

$$\begin{gathered} \tilde{A} = \left(I + \tilde{L}\right)^{-1} + \left(I + \tilde{U}\right)^{-1} + \left(I + \tilde{L}\right)^{-1}\left(\tilde{D} - 2I\right)\left(I + \tilde{U}\right)^{-1}, \\ \tilde{D} = \hat{G}^{-1} D \check{G}^{-1}, \qquad \tilde{L} = \hat{G}^{-1} L \check{G}^{-1}, \qquad \tilde{U} = \hat{G}^{-1} U \check{G}^{-1}, \end{gathered} \tag{23}$$

where $\hat{G}$ and $\check{G}$ are some easily invertible diagonal (or block-diagonal) matrices selected from the condition of approximate minimization of the condition number $cond(\tilde{A})$. For example, good practical results, confirmed in the simple cases by theoretical estimates, are demonstrated by the formula with the diagonal matrices

$$\hat{G} = \check{G} = G^{\frac{1}{2}},\ \ G = \omega^{-1}D,\ \ \omega = b - \sqrt{b^2 - 4ab}/2a,\ \ a = (LD^{-1}Ue, e),\ \ b = (De, e), \qquad (24)$$

where $e = \{1, \dots, 1\}$ is the vector with unit components. The corresponding approach, in according to [3], [12], will be called the incomplete Eisenstat factorization IFE. Its distinguished feature is the efficiency of implementation of each iteration, since the multiplication of the vector by the matrix $\tilde{A}$ according to the formula

$$\tilde{A}v = \left(I + \tilde{L}\right)^{-1}\left[v + \left(\tilde{D} - 2I\right)w\right] + w, \qquad w = \left(I + \tilde{U}\right)^{-1}v \qquad (25)$$

requires almost as many arithmetic operations as multiplications by the original matrix A.

An alternative to the SCR method is the A^TA-orthogonalization algorithm for the direction vectors using three-term Lanczos recurrent formulas

$$\begin{gathered} p^0 = A^Tr^0,\ \ p^1 = A^TAp^0 - \mu_0 p^0, \qquad n = 1,2, \dots : \\ p^{n+1} = A^TAp^n - \mu_n p^n - \nu_n p^{n-1} = A^Tq^n - \mu_n p^n - \nu_n p^{n-1}, \\ q^n = Ap^n, \mu_n = \varkappa_n/\rho_n, \varkappa_n = (A^Tq^n, A^Tq^n), \\ \nu_n = \varkappa_n/\varkappa_{n-1} = \gamma_n/\rho_n, \gamma_n = (A^Tr^n, A^Tr^n). \end{gathered} \qquad (26)$$

Moreover, the vectors u^{n+1}, r^{n+1} are calculated by formulas (15), as in the SCR method. The resulting algorithm will be denoted as CRL, and when using formulas (21) - (25) for preconditioning, as CRL-IFE.

It can be shown that the direction vectors p^n equivalent to the CRL algorithm are realized by the two-term recursions, which formally define the method of conjugate residuals with the preconditioning matrix A^T (we denote it as $\mathrm{CRA^T}$, and in the preconditioned version as $\mathrm{CRA^T}-$IFE):

$$\begin{gathered} r^0 = f - Au^0, \qquad p^0 = A^Tr^0, \\ u^{n+1} = u^n + \alpha_n p^n, \qquad r^{n+1} = r^n - \alpha_n Ap^n, \\ \alpha_n = \gamma_n/\rho_n, \qquad p^{n+1} = A^Tr^{n+1} + \beta_n p^n, \\ \beta_n = \gamma_{n+1}/\gamma_n, \qquad \mathrm{n} = 1,2, \dots \end{gathered} \qquad (27)$$

In all the considered algorithms, the stopping criterion iterations is the fulfillment of the condition

$$(r^n, r^n) \le \varepsilon^2(f, f), \qquad \varepsilon \ll 1. \qquad (28)$$

4. Examples of numerical experiments

We carry out a comparative analysis of the considered algorithms based on the results of an experimental study, focusing on the nature of the dependence of the number of iterations on the dimension of the algebraic system, as well as on the value of the convective coefficients of the original problem.

The calculations were carried out with a standard double precision with the stopping criterion of iterations $\varepsilon = 10^{-7}$ in condition (28). Grid boundary value problems were solved in a square domain with the number of the square grid nodes $N = M^2 = 15^2, 31^2, \dots, 1023^2$. In all examples, the convective coefficients were taken to be constant $q = 0, 1, 2, \dots, 32$, and for the zero value the SLAE matrix is symmetric. The boundary conditions and the right-hand side of the original problem were chosen to correspond to the exact solution $u_e(x, y) = 1$. The initial guess for iterative processes was chosen as $u^0 = 0$.

Table 3 shows the results of calculations using the CRL method (26) without preconditioning. In each cell, the following three numbers are given from top to bottom: the number of iterations n, the final iteration error $\delta = \max_{i,j}\left|u^e_{i,j} - u^n_{i,j}\right|$ and the resulting relative residual $\rho = \|f - Au^n\|_2/\|f\|_2$. As

can be seen from this, the values of δ, ρ are comparable with ε from the stopping condition of iterations, and they are not presented in the subsequent tables.

Table 3. The results of calculations for the CRL method.

q\M	15	31	63	127	255	511	1023
	38	169	619	2309	8826	>10000	>10000
0	1.27e-08	1.1e-08	2.19e-08	5.13e-08	9.74e-08	9.9	15.1
	8.93e-08	6.04e-08	7.53e-08	8.71e-08	9.22e-08	0.00197	0.00199
	118	371	624	902	982	992	994
1	1.9e-08	3.21e-07	8.97e-07	9.71e-07	1.17e-06	1.11e-06	1.11e-06
	4.72e-08	9.85e-08	9.89e-08	9.97e-08	9.95e-08	9.88e-08	9.88e-08
	96	176	235	255	256	253	248
2	3.5e-08	1.78e-07	2.35e-07	2.72e-07	2.68e-07	2.62e-07	2.63e-07
	6.87e-08	9e-08	9.61e-08	9.99e-08	9.65e-08	9.45e-08	9.48e-08
	48	61	65	65	63	62	60
4	4.49e-08	4.85e-08	5.55e-08	5.47e-08	6.31e-08	5.56e-08	6.19e-08
	8.81e-08	8.37e-08	8.83e-08	8.36e-08	9.62e-08	8.48e-08	9.44e-08
	17	18	17	17	16	16	15
8	6.39e-09	5.79e-09	9.11e-09	5.94e-09	8.96e-09	5.78e-09	8.82e-09
	4.88e-08	4.22e-08	6.59e-08	4.27e-08	6.5e-08	4.16e-08	6.45e-08
	6	6	5	5	5	5	5
16	4.76e-10	1.9e-10	1.44e-09	8.9e-10	5.61e-10	3.49e-10	2.1e-10
	2.69e-08	1.07e-08	8.06e-08	4.96e-08	3.13e-08	1.96e-08	1.19e-08
	3	3	3	2	2	2	2
32	2.9e-13	7.03e-14	3.38e-14	2.75e-11	1.91e-11	1.34e-11	9.47e-12
	8.66e-10	2.1e-10	1.01e-10	8.19e-08	5.69e-08	4e-08	2.82e-08

In table 4, we provide a comparison of the number of iterations for the CR-Gauss method — conjugate residuals with a left Gaussian transformation. As can be seen from this, these results are qualitatively close.

Table 4. The results of calculations for the CRA$^{\mathbf{T}}$-IFE method, $\omega = \omega_e$.

q\M	15	31	63	127	255	511	1023
0	39	171	621	2311	8826	>10000	>10000
1	118	349	500	747	812	822	830
2	95	160	219	238	243	243	242
4	48	61	66	66	65	64	63
8	17	18	17	17	16	16	15
16	6	6	5	5	5	5	5
32	3	3	3	2	2	2	2

Next, table 5 presents the results of calculations for the same series of SLAEs of preconditioned matrices $\tilde{A}$ and $\tilde{A}^T$, according to formulas (21), (23), for the CRA$^{\mathrm{T}}$method with the Eisenstat factorization. Here, each cell of the table contains four values — the number of iterations for different values of the relaxation parameter ω. The upper value was obtained at $\omega = 1$, the second and the third from the top correspond to the values $\omega = 1.315$ and $\omega = 1.95$, found by the bisection method to minimize the number of iterations in cells with parameters $M = 15, q = 4$ and $M = 1023, q = 0$, respectively. The lower values are the number of iterations with the parameters $\omega = \omega_e$, calculated by formula (24).

Table 5. The results of calculations for the CRA$^{\mathrm{T}}$-IFE method, $\omega = 1, 1.315, 1.95, \omega_e$.

q\M	15	31	63	127	255	511	1023
0	1: 31 1.31: 23 1.95: 33 1.52: 23	1: 88 1.31: 57 1.95: 61 1.64: 42	1: 288 1.31: 169 1.95: 99 1.73: 82	1: 1045 1.31: 576 1.95: 161 1.8: 179	1: 3894 1.31: 2125 1.95: 285 1.86: 426	1: >5000 1.31: >5000 1.95: 664 1.9: 1064	1: >5000 1.31: >5000 1.95: 1987 1.93: 2750
1	1: 27 1.31: 21 1.95: 33 1.48: 20	1: 59 1.31: 38 1.95: 58 1: 59	1: 93 1.31: 54 1.95: 89 1.62: 30	1: 122 1.31: 66 1.95: 109 1.66: 30	1: 130 1.31: 71 1.95: 114 1.68: 28	1: 132 1.31: 71 1.95: 113 1.69: 26	1: 131 1.31: 70 1.95: 107 1.7: 25
2	1: 20 1.31: 15 1.95: 31 1.38: 15	1: 30 1.31: 19 1.95: 52 1.43: 16	1: 35 1.31: 21 1.95: 74 1.46: 16	1: 38 1.31: 21 1.95: 86 1.47: 15	1: 37 1.31: 21 1.95: 90 1.48: 14	1: 37 1.31: 20 1.95: 87 1.49: 14	1: 36 1.31: 20 1.95: 83 1.49: 13
4	1: 11 1.31: 9 1.95: 25 1.2: 9	1: 12 1.31: 8 1.95: 35 1.22: 9	1: 13 1.31: 8 1.95: 39 1.23: 8	1: 12 1.31: 8 1.95: 40 1.23: 8	1: 12 1.31: 7 1.95: 38 1.23: 8	1: 12 1.31: 7 1.95: 36 1.24: 8	1: 12 1.31: 7 1.95: 34 1.24: 7
8	1: 5 1.31: 7 1.95: 14 1.04: 5	1: 5 1.31: 7 1.95: 14 1.05: 5	1: 5 1.31: 6 1.95: 14 1.05: 5	1: 5 1.31: 6 1.95: 13 1.05: 5	1: 5 1.31: 6 1.95: 13 1.05: 5	1: 5 1.31: 6 1.95: 12 1.05: 4	1: 5 1.31: 5 1.95: 12 1.05: 4
16	1: 3 1.31: 4 1.95: 6 1: 3	1: 3 1.31: 4 1.95: 5 1: 3	1: 3 1.31: 4 1.95: 5 1: 3	1: 3 1.31: 4 1.95: 5 1: 3	1: 2 1.31: 4 1.95: 5 1: 2	1: 2 1.31: 4 1.95: 5 1: 2	1: 2 1.31: 3 1.95: 4 1: 2
32	1: 2 1.31: 2 1.95: 3 1: 2	1: 2 1.31: 2 1.95: 3 1: 2	1: 1 1.31: 2 1.95: 3 1: 1	1: 1 1.31: 2 1.95: 2 1: 1	1: 1 1.31: 2 1.95: 2 1: 1	1: 1 1.31: 2 1.95: 2 1: 1	1: 1 1.31: 2 1.95: 2 1: 1

Table 6 shows the comparative efficiency of the GMRES method, without preconditioning and restarts, i.e. with storage of all direction vectors (results for $M = 1023$ are not presented, since they require a lot of memory).

Table 6. The results of calculations for the GMRES method.

q\M	15	31	63	127	255	511
0	27	57	109	213	413	1120
1	41	64	83	88	89	89
2	34	43	45	45	44	44
4	23	23	22	21	21	20
8	11	10	10	9	9	9
16	5	5	4	4	4	4
32	3	2	2	2	2	2

Finally, table 7 shows data on the number of iterations for the method of conjugate residuals (CR), which are intended, generally speaking, to solve only symmetric SLAEs. In each cell of the table, the upper value is the number of iterations in the CR method without preconditioning, and the lower one with preconditioning at values of the relaxation parameter $\omega = \omega_e$ according to formula (24) for each cell, respectively.

Table 7. The results of calculations for the CR and CR-IFE method, $\omega = \omega_e$.

q\M	15	31	63	127	255	511	1023
0	27 14	57 19	109 28	213 43	413 65	800 101	1541 157
1	80 13	89 15	88 15	88 15	89 14	89 13	88 13
2	121 10	67 10	47 10	45 9	45 9	44 9	43 9
4	2166 7	28 7	22 6	21 6	21 6	20 6	20 6
8	19 4	11 4	10 4	9 4	9 4	9 4	9 4
16	5 2	5 2	4 2	4 2	4 2	4 2	4 2
32	3 1	2 1	2 1	2 1	2 1	2 1	2 1

Comparison of the CRL and GMRES methods without preconditioning shows their similar effectiveness. Since the rate of convergence of iterations in these algorithms is determined by the condition numbers of the matrices A^TA and A, respectively, the number of iterations in GMRES is much less, but its implementation at the n-th step requires computing $O(n)$ scalar and vector additions and products.

The CR method, without preconditioning and with preconditioning, is naturally the most effective for symmetric SLAEs. At a glance, it may seem surprising that this algorithm retains record performance even in the presence of non-zero convection, when $A \neq A^T$. However, the situation in this case is explained by the fact that we present experimental studies for model boundary value problems with separable variables when the matrices $A^x$and $A^y$are commutative. This allows one to symmetrize the original matrix A by a diagonal similarity transformation of the form of (10). And since the condition number A improves with an increase in the convective coefficient q, this only

leads to acceleration of the CR method. This property explains the fast convergence of CR algorithm because spectrum of matrix A is real in our case, see also [16].

5. Conclusion

The above studies of iterative conjugate residuals methods, or minimal residual with orthogonalization of the direction vectors for solving grid diffusion-convection SLAEs, allow us to draw at least two conclusions. First, these approaches are quite competitive in relation to classical methods such as GMRES or SCR. Second, for the considered actual classes of algebraic systems, the "strengthening" of the asymmetry of matrices with an increase in the convection coefficient, as is shown by theory and experiments, only leads to acceleration of the considered iterative processes. A more detailed comparative analysis of the effectiveness of algorithms with the development of recommendations for their applicability in specific circumstances requires additional research.

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