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**GEOPHYSICS**

# The P-Wave Focusing Effect in a Low-Velocity Core of the Earth: Analytical Solution

**A. G. Fatyanov[a] and V. Yu. Burmin[b,*]**

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**Abstract**—This paper reports the calculation data on wave fields for the Earth's model including the mantle and external and internal core with real parameters. The investigation has made it possible to identify physical phenomenon unknown previously such as the formation of the wave field focusing region in the Earth, emerging to the day surface before the first PKP-wave. This is due to the fact that spherical bodies (in this case, the low-velocity core of the Earth) are characterized by the collecting lens properties.

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## INTRODUCTION

The objective of this work is to construct a stable analytical solution for a wave field in a layered sphere with an arbitrary size. After the known Fourier–Legendre transformations, the desired formulation is reduced to a two-parameter family of boundary-value problems for ordinary differential equations. The solution of the unknown problem in each spherical layer is given as a linear combination of Bessel functions [1]. Then, the unknown coefficients are determined from known conjugation conditions at the boundary of the spherical layers. As a result, they are determined through the matrix system of linear equations. For a small number of layers, its solution can be obtained in an explicit form. Due to the fact that different Bessel functions rapidly tend to zero and infinity, the solution is marked by some uncertainty. Moreover, the greater the sphere radius in relative values (wavelengths), the faster this uncertainty arises. In this case, computational calculations become unstable [2]. The solution is constructed using a new asymptotics of the cylindrical functions obtained in [3]. This method yields a stable analytical solution for the wave field in a nonuniform sphere with an arbitrary size.

This paper reports the calculation data on wave fields for the Earth's model including the mantle and external and internal core with real parameters [4]. The investigation has made it possible to identify physical phenomenon such as the formation of the wave field focusing region in the Earth, emerging to the day surface before the first PKP-wave. This is due to the fact that spherical bodies (in this case, the low-velocity core of the Earth) are characterized by the collecting lens properties (for example, [5]).

## STATEMENT OF TASK

The mathematical statement for modeling of P-waves is formulated in a spherical coordinate system $(0 \le r \le R_0,\ 0 \le \theta \le \pi,\ 0 < \varphi \le 2\pi)$ in the following way: determination of the function $u(r,\theta,\varphi,t)$ from the equation:

$$\frac{1}{v^2(r)}\frac{\partial^2 u}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} + f_r f(t) \tag{1}$$

with initial conditions and boundary data:

$$u = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial r}\right|_{r=R_0} = 0. \tag{2}$$

At the boundaries where a longitudinal wave velocity $v$ is discontinuous, the known conjugation conditions are introduced [1]. In (1) and (2), $R_0$ is the sphere

[a] *Institute of Computational Mathematics and Mathematical Geophysics, Siberian Branch, Russian Academy of Sciences, Novosibirsk, 630090 Russia*
[b] *Schmidt Joint Institute of Physics of the Earth, Russian Academy of Sciences, Moscow, 123995 Russia*
**e-mail: burmin@ifz.ru*

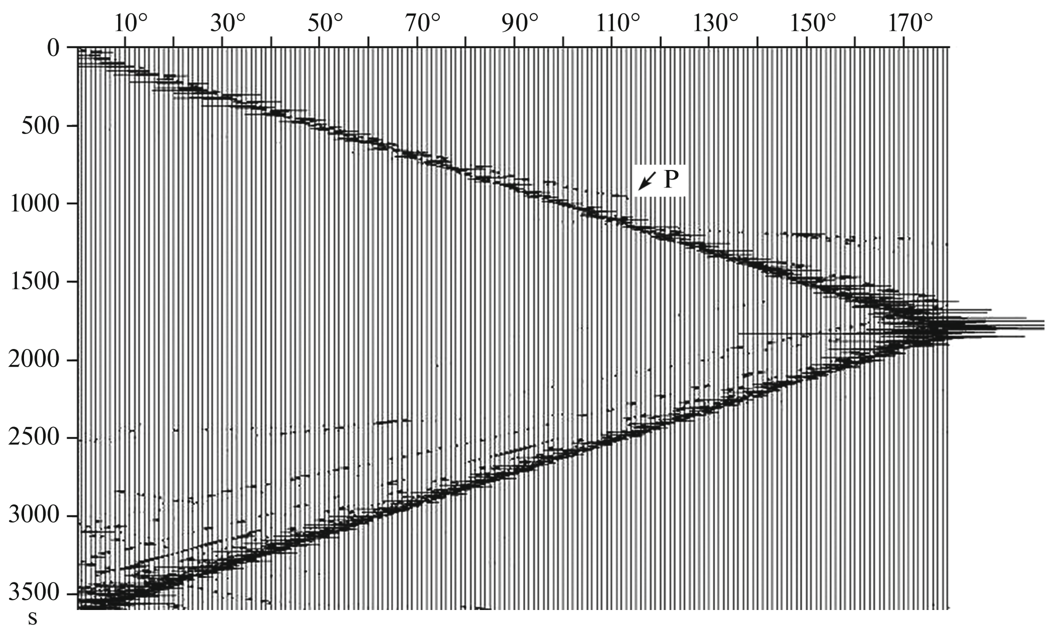


**Fig. 1.** P-wave field in the model of the Earth. The beginning of the blind zone of the direct wave is shown with an arrow.

radius, $f_r$ is the source function, and $f(t)$ is the action of time.

## ANALYTICAL SOLUTION

The concentrated effect $f_r = \delta(r-d)\dfrac{\delta(\theta)}{d^2\sin\theta}$ applied to the points $r = d$ and $\theta = 0$ results in excitation of the displacement field $u(r,\theta,t)$ not dependent on the coordinate φ. At the first stage, the solution is sought as the Fourier–Legendre expansion with respect to variables $(\theta, t)$.

$$u_\varphi(r,\theta,t) = \frac{1}{2T}\frac{1}{\sqrt{r}}\sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty}u(r,k,\omega_n) \times \exp(i\omega_n t)P_k(\cos\theta). \tag{3}$$

Here $P_k(x)$ is a Legendre polynomial and $\omega_n = n\pi/T$. As a result, the formulation (1), (2) is reduced to the two-parameter family ($k$, $\omega_n$) of boundary-value problems in each layer $r_j < r < r_{j+1}$. In order to reduce the record, inessential variables are denoted with a letter $c$, and inessential indexes are omitted.

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{(k+0.5)^2}{r^2}u + c\delta(r-d)F(\omega) = -\frac{\omega^2}{c^2}u, \quad \left.\frac{du}{dr} - \frac{0.5}{r}u\right|_{r=R_0} = 0. \tag{4}$$

At the second stage, the solution in each spherical layer with the number $n$ is determined as a linear combination of Bessel functions [1]:

$$u(r,k,\omega) = c_1^n J_{k+0.5}(\omega r/v) + c_2^n J_{-k-0.5}(\omega r/v). \tag{5}$$

In the internal spherical layer (containing the sphere's core),

$$u(r,k,\omega) = cJ_{k+0.5}(\omega r/v). \tag{6}$$

The unknown coefficients $c_1^n$, $c_2^n$, $c$ are based on the conjugation conditions at the discontinuity boundaries of velocity $v$. These calculations have yielded an explicit solution for the three-layer sphere. It is not given in this paper due to the complexity and obvious nature of the method used to obtain it. Due to the fact that the Bessel functions for positive and negative indexes rapidly tend to zero and infinity, the solution is marked by some uncertainty. Moreover, the greater the sphere radius in relative values (wavelengths), the faster this uncertainty arises. In this case, computational calculations become unstable [2]. Then, the solution is constructed using a new asymptotics of the cylindrical functions, because the common asymptotics yields errors [3]. This method makes it possible to ensure a stable analytical solution for wave fields in the nonuniform sphere with an arbitrary size. Finally, the summation in (3) yields a solution in the physical region.

## RESULTS

Figure 1 shows the obtained calculation data on P-waves for the simplified model of the Earth consisting of a mantle and an external and internal core [4]. The longitudinal wave velocity is 11 km/s in the mantle, 9.5 km/s in the external core corresponding to a liquid state, and 11 km/s in the internal core. The car-

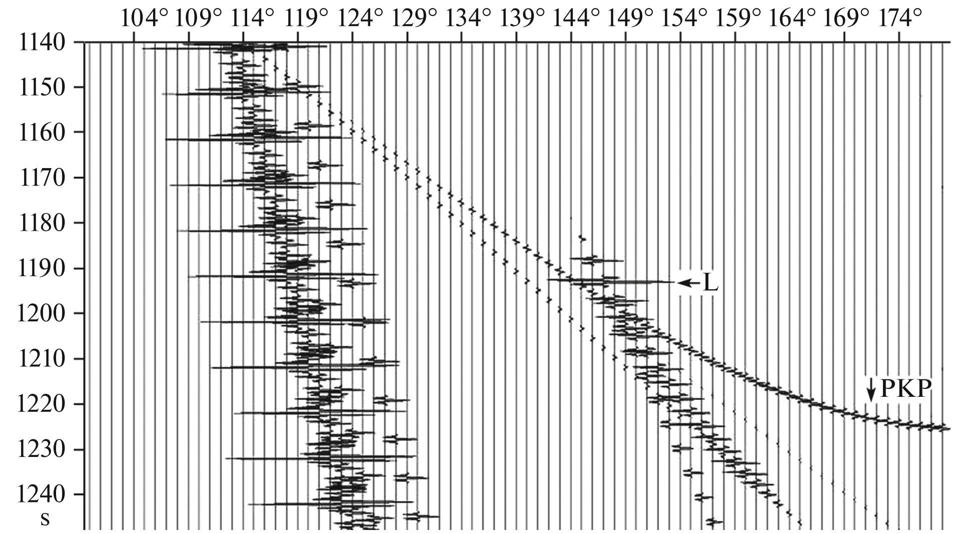


**Fig. 2.** Fragment of Fig. 1 with the first arrival of PKP-waves. Focusing effect of the low-velocity external core (L).

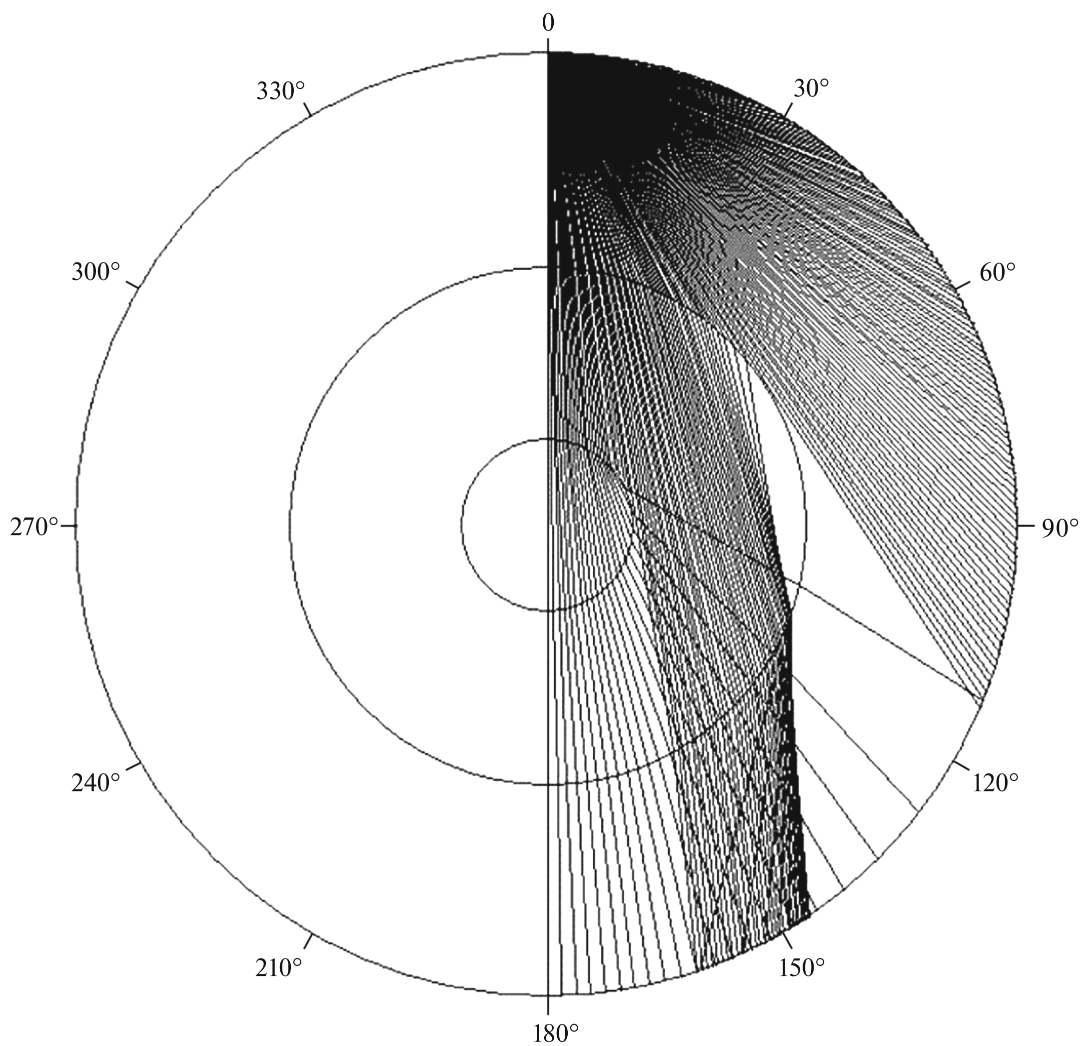


**Fig. 3.** Ray pattern for the model of the Earth. Rays are generated in a regular manner from the source.

rier frequency is assumed to be 1 Hz. The time action $f(t)$ is taken as the Gauss–Puzyrev pulse [3]. Horizontally, the distance is measured in degrees. Vertically, the time is measured in seconds (increases downward). The calculation was carried out for 3600 s (1 h). In Fig. 1, an arrow indicates the beginning of the blind zone for a direct longitudinal P-wave propagating in the Earth's mantle. Figure 2 shows a fragment of the wave field for Fig. 1. Figure 2 demonstrates a considerable amplitude growth indicated as L (Lens), ahead of the PKP-wave passing through the internal core. To illustrate this phenomenon, Fig. 3 shows the ray pattern. As follows from Fig. 3, the external low-velocity core is characterized by the collecting lens properties. Formation of the focus area leads to generation of a powerful wave. It should be noted that this phenomenon is not observed in the high-velocity core.

Why was this phenomenon not detected earlier in the course of the numerical simulation? When modeling by the finite difference algorithms, because of the considerable scope of computations of teleseismic distances, both before [6] and now, the carrier frequency is commonly taken as up to 0.1–0.3 Hz. This condition results in smearing of the wave pattern [7, 8]. However, the analytical modeling makes it possible to take a carrier frequency of 1 Hz or more. It should be noted that this phenomenon is easily detected when solving the direct kinematic problem [4] and is clearly illustrated by construction of the ray pattern in Fig. 3.

## CONCLUSIONS

A stable analytical solution has been obtained for the longitudinal wave fields in the three-layer sphere with an arbitrary size. The total wave fields and rays have been calculated for a simplified model of the Earth consisting of the mantle and external and internal cores with real parameters. According to the mathematical modeling results, the Earth's low-velocity core having the collecting lens features forms the wave field focusing the region inside itself. This process results in oscillations, so-called "precursors," arising on the Earth's surface before the first PKP-wave, which also go on in the following oscillations with a maximum amplitude growth around 147°.

It also should be noted that detection of the focusing region of the oscillations emerging to the surface upon the first arrivals is extremely important in investigation of the state of the core not only in the Earth, but also on other planets, in particular, for the Moon. If we managed to detect the above-mentioned oscillations in the Moon seismograms, the question of the core state and size, as well as of the longitudinal wave velocity in the Moon core, would be solved unambiguously. In the currently considered models of the Moon's internal structure, the oscillations generated by focusing inside the core should emerge at the Moon's surface at a distance of 180°–220° or in the opposite direction in the range of 140°–180°. Unfortunately, such oscillations have not been detected to date in the above-mentioned angle range [9].

## REFERENCES

1. A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics* (Nauka, Moscow, 1977) [in Russian].
2. V. A. Korneev and L. R. Johnson, Geophys. J. Int. **115**, 230–250 (1993).
3. A. G. Fatyanov, Mat. Zametki SVFU **23** (3), 91–103 (2016).
4. V. Yu. Burmin, Izv., Phys. Solid Earth **40** (6), 477–490 (2004).
5. Yu. A. Kravtsov and Yu. I. Orlov, *Geometrical Optics of Inhomogeneous Media* (Nauka, Moscow, 1980) [in Russian].
6. A. G. Fatyanov, Preprint No. 337 (Comput. Center Siberian Branch USSR Acad. Sci., Novosibirsk, 1981).
7. P. Boué, Geophys. J. Int. **194**, 844–848 (2013).
8. V. Yu. Burmin, D. S. Kravtsov, I. V. Luk'yanov, and A. G. Fatyanov, Geofiz. Issled. **15** (2), 5–14 (2014).
9. P. Lognonné, J. Gagnepain-Beyneix, and H. Chenet, Earth Planet. Sci. Lett. **211**, 27–44 (2003).

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