ISSN 1064-5624, Doklady Mathematics, 2020, Vol. 102, No. 3, pp. 478–482.

MATHEMATICS

On Moment Methods in Krylov Subspaces

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Presented by Academician of the RAS E.E. Tyrtyshnikov May 18, 2020

Received June 2, 2020; revised June 2, 2020; accepted November 11, 2020

Abstract—Moment methods in Krylov subspaces for solving symmetric systems of linear algebraic equations (SLAEs) are considered. A family of iterative algorithms is proposed based on generalized Lanczos orthogonalization with an initial vector v^0 chosen regardless of the initial residual. By applying this approach, a series of SLAEs with the same matrix, but with different right-hand sides can be solved using a single set of basis vectors. Additionally, it is possible to implement generalized moment methods that reduce to block Krylov algorithms using a set of linearly independent guess vectors $v_1^0, \ldots, v_m^0$. The performance of algorithm implementations is improved by reducing the number of matrix multiplications and applying efficient parallelization of vector operations. It is shown that the applicability of moment methods can be extended using preconditioning to various classes of algebraic systems: indefinite, incompatible, asymmetric, and complex, including non-Hermitian ones.

Keywords: moment method, Krylov subspace, parametric Lanczos orthogonalization, conjugate direction algorithms

DOI: 10.1134/S1064562420060241

In 1958, Leningrad mathematician Vorob'ev published a book [1] concerning moment methods in Galerkin spaces associated with numerous problems in operator theory and applied mathematics [2]. The results presented in [1] led to a generalization of the conjugate gradient method, which was earlier proposed by K. Lanczos, M. Hestenes, and E. Stiefel for solving systems of linear algebraic equations (SLAEs) (see [3–6]). Specifically, Vorob'ev constructed low-cost algorithms for solving SLAEs with a sign-changing spectrum and different right-hand sides, but with an identical matrix. In 2013, moment methods were investigated and applied in [3] for solving complex SLAEs and reducing models of multiscale dynamical systems.

The goal of this paper is to develop the moment method in Krylov subspaces and generalize it to conjugate direction algorithms [5] and to preconditioned iterative processes intended for solving real SLAEs. The considered algorithms can easily be extended to complex Hermitian systems.

1. POLYNOMIAL REPRESENTATION OF AN APPROXIMATE SOLUTION TO SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

Consider a symmetric positive definite SLAE of the form

$$Au = f, \quad A = A^{\top} = \mathcal{R}^{N,N}, \quad u, f \in \mathcal{R}^{N}, \qquad (1)$$

where, for vectors, we introduced the Euclidean parametrized scalar product

$$(u,v)_{\gamma} \equiv (A^{\gamma}u, v) = u^{\top} \cdot A^{\gamma}v, \quad (u,u)_{\gamma} = \|u\|_{\gamma}^{2},$$

here, γ is equal to 0, 1, or 2.

Given an arbitrary vector v^0, the n-dimensional Krylov subspace is formed by the set of linearly independent vectors $v^{k+1} = Av^k$, $k = 0, 1, \ldots, n-1$:

$$\mathcal{K}_n(v^0, A) = \mathrm{Span}\{v^0, v^1, \ldots, v^{n-1}\}, \qquad (2)$$

so that the element $E_n v^n = Av^{n-1}$ is the projection of v^n onto $\mathcal{K}_n$ and E_n is the corresponding projector in the sense of the introduced scalar product $(\cdot,\cdot)_{\gamma}$.

Consider the following moment problem: given a set of $n+1$ linearly independent elements $v^0, v^1, \ldots, v^n$ of the Hilbert space $\mathcal{R}^N$, construct a linear operator A_n

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defined on the n-dimensional subspace $\mathcal{K}_n$ that satisfies the conditions

$$v^1 = A_n v^0, \dots, v^{n-1} = A_n v^{n-2},\\ E_n v^n = A_n v^{n-1} = A_n^n v^0. \tag{3}$$

Since $E_n v^n$ is the projection of the element $v^n \in \mathcal{R}^N$ onto $\mathcal{K}_n$, the difference $v^n - E_n v^n$ is orthogonal to any element $v^k \in \mathcal{K}_n$. Then, using the representation

$$E_n v^n = -a_0 v^0 - a_1 v^1 - \dots - a_{n-1} v^{n-1}$$

and taking the scalar product of both sides by $A^\gamma v^k$, we obtain the following system for the coefficient vector $a = (a_0, \dots, a_{n-1})^\top$:

$$Ga = w \equiv ((v^n, v^0)_\gamma, \dots, (v^n, v^{n-1})_\gamma)^\top, \tag{4}$$

with a nonsingular Gram matrix $G = \{(v^k, v^l)_\gamma\} \in \mathcal{R}^{n,n}$. In view of this, SLAE (4) uniquely determines a matrix polynomial of degree n annulling the element v^0:

$$P_n(A_n)v^0 = (A_n^n + a_{n-1}A_n^{n-1} + \dots + a_0 I)v^0 = 0, \tag{5}$$

and the roots of the corresponding scalar polynomial $P_n(\lambda)$ are the eigenvalues of the operator A_n. Here and below, the index γ on matrices, vectors, and polynomials is omitted for brevity.

Now we consider a representation of the solution to the equation

$$A_n u^n = f^n; \quad u^n, f^n \in \mathcal{K}_n, \tag{6}$$

with the right-hand side given by

$$f^n = \sum_{k=0}^{n-1} b_k v^k = \sum_{k=0}^{n-1} b_k A_n^k v^0 = F_{n-1}(A_n)v^0,\\ F_{n-1}(\lambda) = b_{n-1}\lambda^{n-1} + b_{n-2}\lambda^{n-2} + \dots + b_0. \tag{7}$$

Taking the scalar product of (7) and $v^0, \dots, v^{n-1}$, yields an SLAE for the coefficient vector:

$$Gb = \overline{f} \equiv ((v^0 f)_\gamma, (v^1 f)_\gamma, \dots, (v^{n-1}, f)_\gamma)^\top,\\ b = (b_0, b_1, \dots, b_{n-1})^\top. \tag{8}$$

The solution of system (6) can be expressed in terms of the polynomial

$$S_{n-1}(A_n) = A_n^{-1}\left[F_{n-1}(A_n) - P_n(A_n)F_{n-1}(0)/P_n(0)\right], \tag{9}$$

As a result, under the condition $P_n(0) = a_0 \neq 0$, the approximate solution of the moment problem can be expressed using the coefficients a_k and b_k of the polynomials as follows:

$$u^n = S_{n-1}(A_n)v^0 = (b_1 - \overline{a}_1)v^0 + \dots\\ + (b_{n-1} - \overline{a}_{n-1})v^{n-2} - \overline{b}_0 v^{n-1},\\ \overline{a}_k = a_k b_0/a_0, \quad \overline{b}_0 = b_0/a_0. \tag{10}$$

2. RECURSIVE SOLUTION OF THE MOMENT PROBLEM

The performance of the algorithm given by formula (10) can be enhanced by applying the Lanczos orthogonalization of the vectors $v^0, v^1, \dots, v^n$, which underlies the conjugate gradient method. We consider this approach in a generalized form leading to conjugate gradient methods [5]. More specifically, applying parametric A^γ-orthogonalization, we construct a set of vectors p^k with properties $(p^k, p^l)_\gamma = (p^k, p^k)_\gamma \cdot \delta_{k,l}$, where $\gamma = 0$, 1, 2 and $\delta_{k,l}$ is the Kronecker delta. The corresponding formulas are given by

$$p^0 = v^0, \quad p^1 = Av^0 - \overline{\alpha}_0^{(\gamma)}v^0; \quad k = 1, 2, \dots:\\ p^{k+1} = (A - \overline{\alpha}_k^{(\gamma)}I)p^k - \overline{\beta}_k^{(\gamma)}p^{k-1},\\ \overline{\alpha}_k^{(\gamma)} = (p^k, p^k)_{\gamma+1}/\|p^k\|_\gamma^2, \quad \overline{\beta}_k^{(\gamma)} = \|p^k\|_\gamma^2/\|p^{k-1}\|_\gamma^2. \tag{11}$$

Here, the vectors p^k are expressed in terms of matrix polynomials related by three-term equalities:

$$p^k = P_k(A)v^0, \quad \beta_{-1}^{(\gamma)} = 0, \quad P_0(A) = 1,\\ P_{k+1}(A) = (A - \overline{\alpha}_k^{(\gamma)}I)P_k(A) - \overline{\beta}_k^{(\gamma)}P_{k-1}(A). \tag{12}$$

Since the vectors $p^0, \dots, p^{n-1}$ form an A^γ-orthogonal basis in $\mathcal{K}_n$ and in view of the properties of the projector E_n and recurrences (12), we have

$$P_n(A_n)v^0 = E_n P_n(A)v^0 = E_n p^n = 0.$$

It follows that the resulting polynomials P_n coincide with those in (5); the roots of the equation $P_n(\lambda) = 0$ are the eigenvalues of the operator A_n; and, in the orthogonal basis $p^0, \dots, p^{n-1}$, the projector from $\mathcal{R}^N$ to $\mathcal{K}_n$ is orthogonal, i.e.,

$$E_n = V_n V_n^\top, \quad E_n^2 = E_n, \quad V_n \in \mathcal{R}^{N,n},$$

where V_n is a matrix with columns being the vectors p^k.

Remark 1. Another generalization of Lanczos orthogonalization was proposed in [7], namely,

$$p^0 = v^0, \quad c_1 p^1 = Bp^0 - \alpha_0 p^0, \quad k = 2, 3, \dots:\\ c_{k+1}p^{k+1} = Bp^k - \alpha_k p^k - \beta_k p^{k-1},\\ \alpha_k = (p^k, CBp^k)/(p^k, Cp^k),\\ \beta_k = (p^{k-1}, CBp^{k-1})/(p^{k-1}, Cp^{k-1}), \tag{13}$$

where B and C are commuting symmetric matrices and c_k are arbitrary nonzero constants. These relations ensure that the vectors are C-orthogonal, i.e.,

$(p^k, Cp^n) = 0$ for $k \neq n$. Formulas (12) follow from (13) for $B = A$ and $C = A^\gamma$. The case $B = AA^\top = C$ for asymmetric SLAEs was considered in more detail in [7].

Using the recurrence properties of the constructed polynomials, we transform the approximate solution u^n from (10). Introducing the notation $R_n(\lambda) = P_n(\lambda)/P_n(0)$, and taking into account (12), for these polynomials, we obtain the recurrence

$$R_{k+1}(\lambda) = (\lambda - \bar{\alpha}_k^{(\gamma)})R_k(\lambda)\,P_k(0)/P_{k+1}(0) - \bar{\beta}_{k-1}^{(\gamma)}\,R_{k-1}(\lambda)P_{k-1}(0)/P_{k+1}(0). \tag{14}$$

From this, introducing the notation

$$\alpha_k = -P_k(0)/P_{k+1}(0),$$
$$\beta_k = \beta_k^{(\gamma)}R_{k-1}(\lambda)\,P_{k-1}^2(0)/P_k^2(0),$$
$$Q_k(\lambda) = -[R_{k+1}(\lambda) - R_k(\lambda)]/\alpha_k\lambda,$$

we derive the two-term recurrences

$$R_{k+1}(\lambda) = R_k(\lambda) - \lambda\alpha_k Q_k(\lambda),\quad Q_{k+1}(\lambda) = R_{k+1}(\lambda) + \beta_k Q_k(\lambda). \tag{15}$$

Considering, along with these polynomials, corresponding elements of the Hilbert space, i.e., $r^k = R_k(A)v^0$ and $q^k = Q_k(A)v^0$, we obtain similar vector relations

$$r^{k+1} = r^k - \alpha_k A q^k,\quad q^{k+1} = r^{k+1} + \beta_k q^k,\quad q^0 = r^0 = v^0. \tag{16}$$

Based on the properties of the above-introduced polynomials, we conclude that the obtained vector sequences have the following orthogonal properties:

$$(r^k, r^{k'})_{\gamma-1} = \sigma_k\delta_{k,k'},\quad \sigma_k = \|r^k\|_{\gamma-1}^2,\quad (q^k, q^{k'})_\gamma = \rho_k\delta_{k,k'},\quad \rho_k = (q^k, q^k)_\gamma. \tag{17}$$

Combining these relations with (16) yields formulas for the iteration parameters:

$$\alpha_k = \sigma_k/\rho_k,\quad \beta_k = \sigma_{k+1}/\sigma_k. \tag{18}$$

Now we find an expression for the approximate solution (10). Since the elements $r^0, \ldots, r^{n-1}$ form an orthogonal basis in $\mathcal{K}_n$, we have

$$f^n \equiv E_n f = F_{n-1}(A)v^0 = \sum_{k=0}^{n-1} r^k (f, r^k)_\gamma / \|r^k\|_\gamma^2,$$

which implies

$$F_n(0) = \sum_{k=0}^{n} (f, r^k)_\gamma R_k(0)/\|r^k\|_\gamma^2 = F_{n-1}(0) + f_n^n,\quad f_n^n = (f, r^n)_\gamma/\|r^n\|_\gamma^2. \tag{19}$$

Then it follows from (10) and (16) that

$$u^{k+1} = A^{-1}[f^{k+1} - F_k(0)r^{k+1}] = A^{-1}[f^k - F_{k-1}(0)r^k + \alpha_k(F_{k-1}(0) + f_k^r)Aq^k] = u^k + \alpha_k\gamma_k q^k,$$

where $\gamma_n = F_n(0)$. Thus, the solution of the moment problem is determined by the two-term relations

$$u^{k+1} = u^k + \alpha_k\gamma_k q^k,\quad \alpha_k = \|r^k\|_\gamma^2/(Aq^k, q^k)_\gamma,$$
$$r^{k+1} = r^k - \alpha_k A q^k,\quad \gamma_k = \gamma_{k-1} + f_k^k, \tag{20}$$
$$q^{k+1} = r^{k+1} + \beta_k q^k,\quad \beta_k = \|r^{k+1}\|_\gamma^2/\|r^k\|_\gamma^2,$$
$$r^0 = q^0 = v^0,\quad \gamma_0 = (f, v^0)_\gamma^2/\|v^0\|_\gamma^2.$$

These formulas simplify if we set $v^0 = f$, in which case $\gamma_0 = \ldots = \gamma_n = 1$. Setting $v^0 = f - Au^0$ for an arbitrary initial vector u^0, at any nth iteration, we have the residual vector $r^n = f - Au^n$, i.e., we obtain conjugate gradient methods [4, 5].

3. SOME PROPERTIES OF THE ITERATIVE MOMENT METHOD

Algorithm (20) efficiently solves problems of interest, such as series of SLAEs with the same matrix A, but with different right-hand sides $f^{(m)}$, $m = 1, \ldots, M$, which are determined sequentially, i.e., a new mth vector $f^{(m)}$ can be computed only after solving the preceding $(m - 1)$th system. The first SLAE is solved using formulas (20), while the subsequent desired vectors $u^{(m)}$ are determined using the found coefficients α_n and vectors q^n. Only the quantities γ_n have to be updated, which considerably reduces the amount of arithmetic operations, especially because of the absence of matrix-vector multiplications. For approximate solutions u^n, instead of (20), we can use their representation in terms of the vectors q^n, which, in view of the orthogonality properties (17), form a basis in $\mathcal{K}_n$ and satisfy the relations

$$A_n q^k = \alpha_k^{-1}[-\beta_{k-1}q^{k-1} + (1 + \beta_k)q^k - q^{k+1}],$$
$$A_n q^0 = \alpha_0^{-1}[(1 + \beta_0)q^0 - q^1], \tag{21}$$
$$A_n q^{n-1} = \alpha_{n-1}^{-1}[-\beta_{n-1}q^{n-2} + (1 + \beta_{n-1})q^{n-1}].$$

Substituting the desired representation of the solution

$$u^n = \sum_{k=0}^{n-1} \nu_k q^k = u^{n-1} + \nu_{n-1}q^{n-1},\quad u^{-1} = 0, \tag{22}$$

into Eq. (6) and using relations (17) and (20), we obtain the expression

$$\sum_{k=0}^{n-1}\nu_k\alpha_k^{-1}[-\beta_{k-1}q^{k-1}+(1+\beta_k)q^k-q^{k+1}]=\sum_{k=0}^{n-1}f_k\,q^k,$$

$$f_k=(f,q_k)_\gamma/\|q_k\|_\gamma^2.$$

Equating the coefficients of identical vectors yields a tridiagonal system of equations for finding ν_k from (22):

$$-\alpha_{k-1}^{-1}\nu_{k-1}+\alpha_k^{-1}(1+\beta_k)\nu_k-\alpha_{k+1}^{-1}\beta_k\nu_{k+1}=f_k,\qquad \nu_{-1}=\nu_n=0,\quad k=0,1,\dots,n-1. \tag{23}$$

It should be emphasized that, in representation (22) for the solution u^n, the coefficients ν_k depend on n and, to find them, we need to solve a sequence of increasing-order systems of the form (23). For this purpose, an efficient approach is provided by the tridiagonal matrix algorithm [5], whose "standard" formulas in the given case have the form

$$\begin{gathered}æ_1=f_1^q/t_1,\quad æ_k=(f_k^q-x_k\,æ_{k-1})s_k,\\ s_k=(t_k-x_k\,æ_{k-1})^{-1},\\ \theta_1=y_1/t_1,\quad \theta_k=y_ks_k,\quad k=1,2,\dots,n-1,\\ \nu_{n-1}=æ_{n-1},\quad \nu_k=\theta_k\nu_{k+1}+æ_k,\\ k=n-2,\dots,1.\end{gathered} \tag{24}$$

Here, for brevity, we introduced the following notation for the coefficients of SLAE (23):

$$t_k=\alpha_k^{-1}(1+\beta_k),\quad x_k=\alpha_{k-1}^{-1},\quad y_k=\alpha_{k+1}^{-1}\beta_k,$$
$$k=0,1,\dots,n-1;\quad x_1=y_{n-1}=0.$$

The quantities $æ_k$ and θ_k for different n are computed using identical recurrences differing only in length. The implementation of (22) requires one component of the solution from (24): $\nu_{n-1}=æ_{n-1}$.

The stability and accuracy of the tridiagonal matrix algorithm (24) for solving the tridiagonal SLAE (23) play a key role in the efficiency of the entire algorithm. Note that the considered iterative process can be extended to more general types of symmetric algebraic systems, such as indefinite, singular, and inconsistent ones, for which MINRES, MINRES-QLP, and other methods in classical Krylov subspaces were previously developed; they use QR and QLP decomposition algorithms for solving tridiagonal SLAEs of type (23) (see [8] and references therein).

The considered iterative process is a generalization of well-known conjugate direction algorithms, since the initial vector ν^0 of the Krylov subspace $\mathcal{K}_n(v^0,A)$ is not related to the SLAE to be solved. However, the moment methods inherit the variational properties of minimum iteration algorithm, or minimum error algorithms [9], as well as the variational properties of conjugate gradient and conjugate residual algorithms [4, 5], since the vectors r^k can be represented in the form

$$r^k=r^0-\alpha_0Aq^0-\alpha_1Aq^1-\ldots-\alpha_{k-1}Aq^{n-1}, \tag{25}$$

which follows from (20); here, the vectors have orthogonality properties (17). Taking the scalar product of both sides of (25) by the vector $A^{\gamma-2}r^k$ and using the values of α_k from (20), we derive the expressions

$$\begin{gathered}\varphi_\gamma(r^k,r^k)=(r^k,r^k)_{\gamma-2}\\ =(r^0,r^0)_{\gamma-2}-\sum_{l=0}^{k-1}(r^0,q^l)_{\gamma-1}^2\big/(q^l,q^l)_\gamma,\end{gathered} \tag{26}$$

moreover, the maximum of this functional is reached in the subspace $\mathcal{K}_k(v^0,A)$.

In view of the optimality properties of functional (26), we can use the Chebyshev technique to examine the convergence rate of the iteration process and to estimate the number of iterations $n(\varepsilon)$ necessary for satisfying the condition

$$\varphi_\gamma(r^n,r^n)\le\varepsilon^2\varphi_\gamma(r^0,r^0),\quad \varepsilon\ll 1. \tag{27}$$

Obviously, the corresponding inequality has the form

$$n(\varepsilon)\le 1+0.5æ\ln|\varepsilon/2|, \tag{28}$$

where æ is the condition number of the matrix A (see [4, 5]).

Let us discuss the use of preconditioning of SLAEs in moment methods in order to reduce the number æ in estimate (28). To preserve symmetry, it is natural to use bilateral preconditioning, which brings Eq. (1) to the form

$$\bar A\bar u=\bar f,\quad \bar A=C^\top AC,\quad \bar u=C^{-1}u,\quad \bar f=C^\top f, \tag{29}$$

where C is an easily invertible nonsingular matrix. In this case, all preceding arguments hold for the preconditioned SLAE up to the notation of $\bar u,\bar f,\bar A$. An example is incomplete factorization methods [5] for systems with a matrix $A=D+L+U$, where D is a block diagonal matrix and L and U are lower and upper triangular matrices. The preconditioned SLAE is then given by

$$\begin{gathered}\bar A\bar u=(I+\bar L)^{-1}(\bar D+\bar L+\bar U)(I+\bar U)^{-1}\bar u\\ =\bar f=(I+\bar L)^{-1}G_L^{-1}f,\\ \bar L=G_L^{-1}LG_U^{-1},\quad \bar D=G_L^{-1}DG_U^{-1},\\ \bar U=G_L^{-1}UG_U^{-1},\quad \bar u=(I-\bar U)G_Uu,\\ G_LG_U=G=D-\overline{LG^{-1}U},\end{gathered} \tag{30}$$

where $\overline{LG^{-1}U}$ is an approximation of the matrix $LG^{-1}U$. The preconditioners in (30) are a special case of widely used ILU decompositions (see [4, 5]).

Given a symmetric positive definite preconditioner B, the construction of a preconditioned Lanczos process is regarded as an orthogonalization method applied to the preconditioned SLAE

$$\bar{A}\bar{u} = \bar{f} = B^{-1/2}f, \quad \bar{A} = B^{-1/2}AB^{-1/2}, \quad \bar{u} = B^{1/2}u.$$

In formulas (11), for the new vectors $w^k = B^{-1/2}p^k$, we then obtain the recurrence relation

$$w^{k+1} = AB^{-1}w^k - \bar{\alpha}_k^{(\gamma)}w^k - \bar{\beta}_k^{(\gamma)}w^{k-1},$$

in which $B^{1/2}$ is, in fact, not required (for more details, see [8]).

Remark 2. Note also the utility of simple preconditioning procedures, such as scaling or binormalization of SLAE [10], which correspond to diagonal matrices C in (27) and can be used in combination with other preconditioners.

Remark 3. In [1] Vorob'ev considered a generalized moment method in which a single initial vector v^0 is replaced by a given set of m linearly independent vectors $v_1^0, \ldots, v_m^0$. In solution methods for SLAEs, such approaches are promising in terms of parallelization and are known as block Krylov algorithms (see [11]). The methods described above are easy to generalize to their block versions. The resulting algorithms formally admit multipreconditioning, i.e., the use of several preconditioning matrices and direction vectors $p_1^k, \ldots, p_m^k$ at a single iteration (see [12]). However, the justification of this approach for symmetric SLAEs requires additional research.

FUNDING

This work was supported by the Russian Foundation for Basic Research, project no. 18-01-00295.

REFERENCES

1. Yu. V. Vorob'ev, *Method of Moments in Applied Mathematics* (Fizmatgiz, Moscow, 1958; Gordon and Breach, New York, 1965).
2. N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis* (Fizmatgiz, Moscow, 1961; Oliver and Boyd, Edinburgh, 1965).
3. J. Liesen and Z. Strakos, *Krylov Subspace Methods: Principles and Analysis* (Oxford Univ. Press, Oxford, 2013).
4. Y. Saad, Iterative Methods for Sparse Linear Systems, 2nd ed. (SIAM, Philadelphia, 2003).
5. V. P. Il'in, *Finite Element Methods and Technologies* (Inst. Vychisl. Mat. Mat. Geofiz. Sib. Otd. Ross. Akad. Nauk, Novosibirsk, 2007) [in Russian].
6. M. A. Olshanskii and E. E. Tyrtyshnikov, *Iterative Methods for Linear Systems Theory and Applications* (SIAM, Philadelphia, 2014).
7. E. G. Craig, J. Math. Phys. **8**, 64–73 (1955).
8. S.-C. T. Choi, C. C. Paige, and M. A. Saunders, "MINRES-QLP: Krylov subspace method for indefinite or singular symmetric systems," SIAM. arXiv^1003.4042[math.NA]. March 27, 2015.
9. V. M. Fridman, USSR Comput. Math. Math. Phys. **2** (2), 341–342 (1962).
10. O. E. Livne and J. H. Golub, Numer. Algorithms **35** (1), 97–120 (2004).
11. A. A. Nikishin and A. Y. Eremin, SIAM J. Matrix Anal. Appl. **16**, 1135–1153 (1995).
12. V. P. Il'in, Lect. Numer. Comput. Sci. **10187**, 95–107 (2017).

Translated by I. Ruzanova