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A combined inverse tsunami problem*

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Abstract. It is known that some of the parameters required for the direct simulation of tsunamis are the bottom relief characteristics and the initial perturbation data (a tsunami source). The seismic data about the source are usually obtained in a few tens of minutes after an event has occurred (the seismic waves velocity being about five hundred kilometers per minute, while the velocity of tsunami waves is less than twelve kilometers per minute). A difference in the arrival times of seismic and tsunami waves can be used when operationally refining the tsunami source parameters and modeling expected tsunami waves on the shore. The most suitable physical models related to the tsunamis simulation are based on the shallow water equations. We investigate three different inverse problems of determining a tsunami source using three different additional data: DART measurements, satellite wave-form images and seismic data. We investigate a gradient-type and SVD inverse problem solution and show that using a combination of three different types of data allows one to increase the stability and convergence of numerical inverse problem solution. Results of numerical experiments of the tsunami source reconstruction are presented and discussed.

1. Introduction

Tsunamis are gravitational, i.e., gravity-controlled waves resulting from abrupt large-scale perturbations arising during seaquakes, underwater volcano eruptions, underwater landslides, rock fragment falls, underwater explosions, etc. More than 250 tsunamis were observed in the 20th century, about 90 percent of all of them are generated by seaquakes. Hence, a central component of the early warning system is quick detection and evaluation of earthquakes. For this aim, various ocean measuring instruments positioned on the ocean floor, on buoys or in the form of tide gauges are used to recognize an approaching tsunami. The recent severe tsunamis occurred in Japan (2011), Sumatra (2004), and on the Indian coast (2004) have shown that a system giving exact and immediate information about tsunamis is of vital importance. Mathematical modeling and numerical simulations are the most used instruments for providing such an information. It is known that some of the parameters required for direct simulation of tsunamis are bottom relief characteristics and initial perturbation data (a tsunami source). The seismic data concerning the source are usually obtained in a few tens of minutes

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after the event, (the seismic waves velocity being about five hundred kilometers per minute, while the velocity of tsunami waves is less than twelve kilometers per minute). A difference in the arrival times of seismic and tsunami waves can operationally be used when refining the tsunami source parameters and modeling expected tsunami waves on the shore. Most suitable physical models related to simulation of tsunamis are based on shallow water equations (see, [1–3] and references therein). For the tidal motion, even a very deep ocean may be considered as shallow as its depth will always be much smaller than the tidal wave length [1, 4]. The shallow water equations (also called the Saint Venant equations in their uni-dimensional form) are a set of hyperbolic partial differential equations that describe a flow below the pressure surface in a fluid. The equations of shallow water theory are based on the following assumptions: the vertical acceleration of liquid particles is inessential versus the acceleration of gravity, and the horizontal velocities depend but slightly on the vertical coordinate. In this case, the vertical velocity component (versus the horizontal components) may be ignored.

In the Cartesian coordinate system, we formulate the initial boundary-value problem

$$\begin{aligned}
&\eta_t + (Hu)_x + (Hv)_y = 0, \\
&u_t + g\eta_x = 0, \\
&v_t + g\eta_y = 0, \quad (x, y, t) \in \Omega_T := \Omega \times (0, T) \subset \mathbb{R}^3, \;\; T > 0; \\
&\eta(x, y, 0) = q(x, y), \quad u(x, y, 0) = 0, \quad v(x, y, 0) = 0, \;\; (x, y) \in \Omega \subset \mathbb{R}^2,
\end{aligned} \tag{1}$$

for the linear equations of shallow water theory in terms of the liquid flow components in the dimensional form; $\Omega := \{(x, y) \in \mathbb{R}^2 : \; x \in (0, L_x), \; y \in (0, L_y), \; L_x, L_y > 0\}$ is assumed to be a rectangular domain. It is assumed that the action of external forces, e.g., the Coriolis force and the bottom friction, are zero. Note that most of the benchmark problems use no bottom friction. Here $\eta(x, y, t)$ defines the free water surface vertical displacement, i.e., the amplitude of a tsunami wave, the function $H(x, y) > 0$ describes the bottom relief (bathymetry), $u(x, y, t)$ and $v(x, y, t)$ are the depth-averaged velocities in the Ox and Oy directions, respectively, $g = 9.8$ m/s^2 is the acceleration of gravity. Further, $c(x, y) = \sqrt{gH(x, y)}$ is the tsunami propagation velocity, according to the long-wave theory the propagation velocity for tsunami waves, and $q(x, y)$ is the amplitude of a tsunami wave. It is assumed that $q(x, y)$ is a finite function with the compact support $\operatorname{supp} q \subset \Omega$.

Differentiating the first equation in system (1) with respect to the time variable $t > 0$, then using the second and the third equations of this system, we obtain the following initial boundary value problem in terms of the function $\eta(x, y, t)$ for the second order hyperbolic equation:

$$\begin{aligned} &L\eta := \eta_{tt} - \operatorname{div}(c^2(x,y)\operatorname{grad}\eta) = 0, && (x,y,t) \in \Omega_T;\\ &\eta(x,y,0) = q(x,y), \quad \eta_t(x,y,0) = 0, && (x,y) \in \Omega,\\ &\eta(x,y,t) = g(x,y,t), && (x,y,t) \in \partial\Omega \times (0,T). \end{aligned} \tag{2}$$

Here $g(x,y,t)$ is a smooth function defined on the boundary.

With respect to the admissible initial perturbations $q(x,y)$ and the coefficient $H(x,y)$, we will assume that

$$q \in H^1(\Omega), \quad \operatorname{supp} q \subset \Omega, \quad H \in L_\infty(\overline{\Omega}). \tag{3}$$

For a given initial perturbation $q(x,y)$, the hyperbolic problem (2) is defined to be a *direct* (*or forward*) *problem*. The solution of the direct problem will be defined as $\eta(x,y,t;q) \in C[0,T] \cup H^1(\Omega)$, in order to indicate its dependence on the the initial perturbation $q(x,y)$.

2. The DART data as additional information for the inverse problem

One of the tsunami inversion techniques is mainly used to reconstruct the properties of tsunamigenic sources from tsunami records on the set of points $\{(x_m,y_m) \in \Omega,\ m = 1,2,\dots,M,\ M \in \mathbb{N}\}$ (a discrete set of measured output data) or along some smooth simple line $\gamma(s) := (x(s),y(s))$, $s \in (0,1)$, during the time $t \in (T_{1,m},T_{2,m}) \subset (0,T)$ (Deep-Ocean Assessment and Reporting of Tsunamis, DART). This inverse problem is considered in [5, 6].

We assume that the free surface oscillation data at (x_m,y_m) are given as measured output data:

$$f_m(t) := \eta(x_m,y_m,t)\chi_m(t), \quad (x_m,y_m) \in \Omega,\ \ m = 1,2,\dots,M, \tag{4}$$

or, equivalently, $f_\gamma(t) := \eta(x(s),y(s),t)\chi_m(t)$, $(x(s),y(s)) \in \gamma(s)$, $s \in (0,1)$. Here $\chi_m(t)$ is a characteristic function of the interval $(T_{1,m},T_{2,m})$. Without loss of generality we assume that $(T_{1,m},T_{2,m}) \cap (T_{1,k},T_{2,k}) = \emptyset$, for all $m \neq k$, $m,k = 1,2,\dots,M$.

The ocean domain in question is bounded from above by the free surface $\eta(x,y,t)$, and from below—by the bottom relief $H(x,y)$. The lateral boundary $\Gamma_T := \partial\Omega \times (0,T)$ is assumed to be a non-reflecting boundary, that is, it allows a free passage of propagating waves. Thus, the following boundary conditions are assumed at the lateral boundary $\Gamma_T = \overline{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}$, $\Gamma_i \cap \Gamma_j \neq 0$, $i \neq j$, of the domain Ω_T (the type of boundary condition in (2) instead of $g(x,y,t)$:

$$\begin{aligned} &\eta_t - c(x,y)\eta_x = 0, && (x,y,t) \in \Gamma_1 := \{(x,y,t) \in \Gamma_T:\ x = 0\},\\ &\eta_t + c(x,y)\eta_x = 0, && (x,y,t) \in \Gamma_2 := \{(x,y,t) \in \Gamma_T:\ x = L_x\},\\ &\eta_t - c(x,y)\eta_y = 0, && (x,y,t) \in \Gamma_3 := \{(x,y,t) \in \Gamma_T:\ y = 0\},\\ &\eta_t + c(x,y)\eta_y = 0, && (x,y,t) \in \Gamma_4 := \{(x,y,t) \in \Gamma_T:\ y = L_y\}. \end{aligned} \tag{5}$$

The *inverse problem* consists in determining the unknown initial perturbation $q(x, y)$ of the free surface in (2), (5) from the free surface oscillation data $f_m(t)$ given by (4) (or $f_\gamma(t)$).

Lemma 1. *Let conditions* (3) *hold. Then the input-output map* $A : q \in H^1(\Omega) \mapsto F \in C(0, T; E^M)$ *is a linear compact operator. Here* $F(t) := (f_1(t), f_2(t), \dots, f_M(t)) \in E^M$ *is the vector of discrete measured output data,* E^M *is the Euclidean space of the time-dependent observations.*

Since linear equations with compact operators are always ill-posed, the inverse source problem (2), (5), (4) is ill-posed as well [6].

2.1. A variational formulation of the inverse problem. A gradient formula for the data misfit functional

Introduce the data misfit functional

$$J(q) = \|Aq - F\|^2_{L_2(0,T)} := \sum_{m=1}^{M} \int_{T_{1,m}}^{T_{2,m}} [\eta(x_m, y_m, t; q) - f_m(t)]^2 \, dt.$$

Assume that $q, q + \Delta q \in H^1(\Omega)$ and $\Delta\eta(x, y, t; \Delta q) := \eta(x, y, t; q + \Delta q) - \eta(x, y, t; q)$, where $\eta(x, y, t; q) \in C[0, T] \cup H^1(\Omega)$ is the solution of the direct problem (2), (5) corresponding to the given source term $q \in H^1(\Omega)$. Obviously, the function $\Delta\eta := \Delta\eta(x, t, y; \Delta q)$ satisfies the following hyperbolic problem:

$$\begin{aligned}
&\Delta\eta_{tt} - \operatorname{div}(c^2(x, y) \operatorname{grad} \Delta\eta) = 0, \quad (x, y, t) \in \Omega_T;\\
&\Delta\eta(x, y, 0) = \Delta q(x, y), \quad \Delta\eta(x, y, 0)_t = 0, \quad (x, y) \in \Omega;\\
&\Delta\eta_t - c(x, y)\Delta\eta_x = 0, \quad (x, y, t) \in \Gamma_1,\\
&\Delta\eta_t + c(x, y)\Delta\eta_x = 0, \quad (x, y, t) \in \Gamma_2,\\
&\Delta\eta_t - c(x, y)\Delta\eta_y = 0, \quad (x, y, t) \in \Gamma_3,\\
&\Delta\eta_t + c(x, y)\Delta\eta_y = 0, \quad (x, y, t) \in \Gamma_4.
\end{aligned} \tag{6}$$

Consider now the variation $\Delta J(q) := J(q + \Delta q) - J(q)$ of the data misfit functional $J(q)$. We have

$$\begin{aligned}
\Delta J(q) := &\sum_{m=1}^{M} \int_{T_{1,m}}^{T_{2,m}} 2[\Delta\eta(x_m, y_m, t, \Delta q) - f_m(t)]\Delta\eta(x_m, y_m, t, \Delta q) \, dt + \\
&\sum_{m=1}^{M} \int_{T_{1,m}}^{T_{2,m}} [\Delta\eta(x_m, y_m, t, \Delta q)]^2 \, dt.
\end{aligned}$$

Lemma 2 [6]. *Let conditions* (3) *hold. Assume that* $\eta(x,y,t;q) \in C[0,T] \cup H^1(\Omega)$ *and* $\Delta\eta(x,y,t;\Delta q) \in C[0,T] \cup H^1(\Omega)$ *are the solutions of problems* (2), (5) *and* (6) *for given initial sources* $q, \Delta q \in H^1(\Omega)$, *respectively. Denote by* $\eta(x_m,y_m,t;q)\chi_m(t)$ *and* $\Delta\eta(x_m,y_m,t;\Delta q)\chi_m(t)$, *the corresponding output data. Then the following integral identity* (*relationship*) *holds*:

$$
\begin{aligned}
2\sum_{m=1}^{M}\int_{T_{1,m}}^{T_{2,m}} &[\eta(x_m,y_m,t,q) - f_m(t)]\Delta\eta(x_m,y_m,t,\Delta q)\,dt \\
&= \int_0^{L_x}\int_0^{L_y} \Delta q(x,y)\psi_t(x,y,0)\,dx\,dy, \qquad (7)
\end{aligned}
$$

for all $q, \Delta q \in H^1(\Omega)$, *where* $\psi(x,y,t) \in C[0,T] \cup H^1(\Omega)$ *is the solution of the following final data hyperbolic problem*:

$$
\begin{aligned}
&L\psi = -2\sum_{m=1}^{M}\big\{[\eta(x_m,y_m,t;q) - f_m(t)]\delta(x-x_m)\delta(y-y_m)\chi_m(t)\big\}, \\
&\qquad\qquad\qquad\qquad (x,y,t) \in \Omega_T; \\
&\psi(x,y,T) = 0, \quad \psi_t(x,y,T) = 0, \quad (x,y) \in \Omega; \\
&\psi_t + c(x,y)\psi_x = 0, \quad (x,y,t) \in \Gamma_1, \\
&\psi_t - c(x,y)\psi_x = 0, \quad (x,y,t) \in \Gamma_2, \\
&\psi_t + c(x,y)\psi_y = 0, \quad (x,y,t) \in \Gamma_3, \\
&\psi_t - c(x,y)\psi_y = 0, \quad (x,y,t) \in \Gamma_4.
\end{aligned}
$$

where $L\psi := \psi_{tt} - \operatorname{div}(c^2(x,y)\operatorname{grad}\psi)$.

Note, that in the right-hand side of integral identity (7) represents the inner product of $J'q$ and Δq in the space $L_2(\Omega)$, that is $J'q = \psi_t(x,y,0)$.

2.2. The degree of ill-posedness of the inverse problem

In this section we study the degree of ill-posedness of the inverse problem for the one-dimensional bottom function $H(x)$ $c(x,y) \equiv c(x)$, respectively) using SVD.

Using the finite Fourier series expansion of $\eta(x,y,t)$ and $q(x,y)$, we have N initial boundary value problems for each Fourier coefficient ($n = 1,\dots,N$):

$$
\begin{aligned}
&\eta_{n,tt} = (c^2(x)\eta_{n,x})_x - n^2c^2(x)\eta_n, && x \in (0,L_x), \quad t \in (0,T); \\
&\eta_n(x,0) = q_n(x), \quad \eta_{n,t}(x,0) = 0, && x \in (0,L_x); \\
&(\eta_{n,t} - c(x)\eta_{n,x})|_{x=0} = 0, && t \in (0,T); \\
&(\eta_{n,t} + c(x)\eta_{n,x})|_{x=L_x} = 0, && t \in (0,T).
\end{aligned} \qquad (8)
$$

$$\eta_n(x_m, t)\chi_m(t) = f_m(t), \quad x_m \in (0, L_x), \quad m = 1, 2, \ldots, M. \tag{9}$$

Using an explicit finite difference scheme, we approximate and represent problem (8), (9) in the algebraic form

$$\boldsymbol{A}(n)Q_n = F. \tag{10}$$

Here $Q_n = (q_{n,0}, q_{n,1}, \ldots, q_{n,N_x})^T$, $F = (f_1^k, \ldots, f_M^k)^T$. We use the approach given in [10] to describe in brief the algorithm of constructing the matrix $\boldsymbol{A}(n)$. To this end, we introduce additional notations.

Let h_t be a time step, $\eta_{i,j}^k$ be the approximate value of $\eta(x_i, y_j, t_k)$ at the mesh points (x_i, y_j, t_k) of a uniform mesh $\omega_h = \big\{(x_i, y_j, t_k) \in \Omega_T : \ x_i = ih_x,\ i = \overline{0, N_x},\ y_j = jh_y,\ j = \overline{0, N_y},\ t_k = kh_t,\ k = \overline{0, N_t}\big\}$. Here $h_x = L_x/N_x$ and $h_y = L_y/N_y$ are the steps along x and y axes, respectively.

Denote

$$v_1 = (\eta_{n,0}^2, \eta_{n,1}^2, \ldots, \eta_{n,N_x}^2)^T, \quad v_2 = (\eta_{n,0}^3, \eta_{n,1}^3, \ldots, \eta_{n,N_x}^3)^T, \quad r = \frac{h_t}{h_x}.$$

Let $U^1 = (v_1, v_2)^T$ and $U^0 = (\eta_{n,0}^0, \eta_{n,1}^0, \ldots, \eta_{n,N_x}^0, \eta_{n,0}^1, \eta_{n,1}^1, \ldots, \eta_{n,N_x}^1)^T$ be $2l$ vectors, $l = N_x + 1$. The vectors v_1 and v_2 can be represented in the form: $v_1 = \boldsymbol{B}_1 U^0$, $v_2 = \boldsymbol{B}_2 v_1 + \boldsymbol{B}_3 U^0$. Here $\boldsymbol{B}_1 = (\boldsymbol{I}^{(N_x-1)} \,|\, \boldsymbol{B}_2)$ is $l \times 2l$ matrix, $\boldsymbol{B}_2$ is $l \times l$ matrix and $\boldsymbol{B}_3 = (\boldsymbol{O}^{(l)} \,|\, \boldsymbol{I}^{(N_x-1)})$ is $l \times 2l$ matrix, $\boldsymbol{O}^{(l)}$ is 0th $l \times l$ matrix and $\boldsymbol{I}^{(N_x-1)} = \mathrm{diag}(0, 1, 1, \ldots, 1, 0)$ is $l \times l$ matrix with $N_x - 1$ non-zero values; $\boldsymbol{B}_2$ is the three-diagonal matrix with entries

$$b_{2,ij} = \begin{cases} 1 - rc_0, & i = j = 0, \\ rc_0, & i = 0,\ j = 1, \\ r^2 \dfrac{c_i^2 + c_{i-1}^2}{2} + h_t n^2 \dfrac{c_{i+1}^2 + c_{i-1}^2}{4}, & 0 < i < N_x,\ j = i - 1, \\ 2 - r^2 \left(\dfrac{c_i^2 + c_{i-1}^2}{2} + \dfrac{c_i^2 + c_{i+1}^2}{2} \right), & 0 < i < N_x,\ j = i, \\ r^2 \dfrac{c_i^2 + c_{i+1}^2}{2} - h_t n^2 \dfrac{c_{i+1}^2 + c_{i-1}^2}{4}, & 0 < i < N_x,\ j = i + 1, \\ rc_{N_x}, & i = N_x,\ j = i - 1, \\ 1 - rc_{N_x}, & i = j = N_x. \end{cases}$$

Theorem 1 [6]. *Let $\boldsymbol{P} = (\boldsymbol{I}^{(l)} \,|\, \boldsymbol{B}_2)$, $\boldsymbol{C} = \begin{pmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2\boldsymbol{B}_1 + \boldsymbol{B}_3 \end{pmatrix}$. Denote by c_k^p the k-th row of the matrix $\boldsymbol{C}^p$ and introduce $M(N_t + 1) \times 2l$-matrix*

$$\boldsymbol{M} = \begin{pmatrix} c^0_{s_m} \\ c^0_{s_m+l} \\ c^1_{s_m} \\ c^1_{s_m+l} \\ \vdots \\ c^{(N_t-1)/2}_{s_m} \\ c^{(N_t-1)/2}_{s_m+l} \end{pmatrix},$$

where s_m is x-coordinate of m-th sensor in the mesh. Then $\boldsymbol{A}(n) = \boldsymbol{MP}$.

We use SVD of the matrix $\boldsymbol{A}(n) = U(n)\Sigma(n)V^T(n)$ to solve problem (10). Here $U(n)$ and $V(n)$ are square orthogonal $M(N_t+1)$ and l matrices, respectively, $\Sigma(n) = \mathrm{diag}(\sigma_0(n), \sigma_1(n), \ldots, \sigma_p(n))$ is $M(N_t+1) \times l$ matrix, $p = \min\{M(N_t+1), l\}$, $\sigma_0(n) > \sigma_1(n) > \ldots > \sigma_p(n) \geq 0$ are singular values of $\boldsymbol{A}(n)$.

Due to the ill-conditioning of the matrix $\boldsymbol{A}(n)$, the solution to system (10) is very sensitive to small perturbations in the measured data F; SVD of the matrix provides insight into the ill-posedness of the original problem.

Figure 1 shows a plot of $\{\sigma_n\}_{n=1}^{600}$ in the logarithmic scale for the matrix $\boldsymbol{A}(n)$ ($n = 1, 5, 15$) for the bottom function

$$H(x,y) = H(x) = 1500 \sin \frac{2\pi x}{L_x} + \frac{H_{\max} - H_{\min}}{L_x} x + H_{\min}.$$

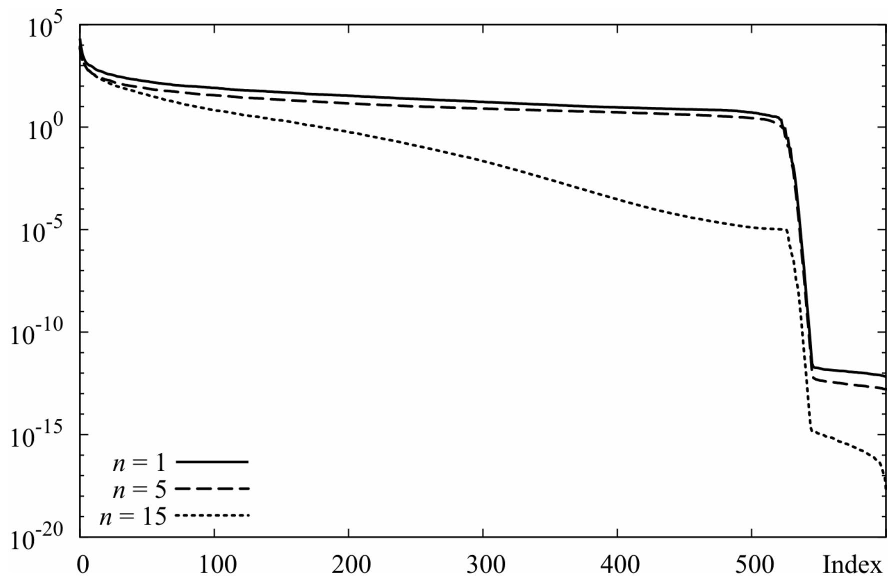

Figure 1. Singular values of the matrix $\boldsymbol{A}(n)$, $n = 1, 5, 15$.

Here $L_x = 50000$ m, $H_{\max} = 6000$ m, $H_{\min} = 5$ m are the highest and the lowest average depths of the ocean, respectively.

From Figure 1 it is evident that the inverse problem (8), (9) is severely ill-posed, i.e., $\sigma_n = O(e^{-n\alpha})$ for some $\alpha > 0$ [7].

3. Satellite data as additional information for the inverse problem

Let us consider the problem of determining the water surface oscillations in a domain after perturbations of the bottom of the domain at $t = 0$ described by the function $f^{(1)}(x, y) = u(x, y, 0)$. It is assumed that in a certain period of time $t = T$ the shape of the water surface is measured and found to be $f^{(2)}(x, y) = u(x, y, T)$. This inverse problem is motivated by altimetry measurements from satellite [8]. Altimeters have been designed to observe the sea level variability. The satellite altimetry contribution is for a better understanding and improving the quality of the modeling of tsunami propagation and dissipation.

We also assume that the time period T is not sufficiently long for a wave to reach the edges of the domain, and therefore we can set homogeneous boundary conditions at the boundary of the domain. Thus, we arrive at the following Dirichlet problem for the wave equation:

$$\begin{aligned} &L\eta = 0, && (x, y, t) \in \Omega_T, \\ &\eta|_{t=0} = f^{(1)}(x, y), \quad \eta|_{t=T} = f^{(2)}(x, y), && (x, y) \in \Omega, \\ &\eta|_{\partial\Omega} = 0, && t \in (0, T). \end{aligned} \tag{11}$$

For the conditions on $\partial\Omega$ to be homogeneous, we require that the support of the function $f^{(1)}(x, y)$ be sufficiently small:

$$\operatorname{supp} f^{(1)} \subset \Omega(a) = \Big(\frac{b}{2} - a, \frac{b}{2} + a\Big) \times \Big(\frac{b}{2} - a, \frac{b}{2} + a\Big), \\ a \in (0, b/2), \quad b = \min(L_x, L_y);$$

as well as the parameter T:

$$T \in (0, T_{\max}), \quad \text{where} \quad T_{\max} = \frac{b/2 - a}{\|c\|_{C(0,b)}}.$$

We now formulate the ill-posed problem (11) as an inverse problem with respect to the direct (well-posed) initial boundary value problem for the wave equation

$$\begin{aligned}
&L\eta = 0, && (x,y,t) \in \Omega_T, \\
&\eta|_{t=0} = f^{(1)}(x,y), \quad \eta_t|_{t=0} = q(x,y), && (x,y) \in \Omega, \\
&\eta|_{\partial\Omega} = 0, && t \in (0,T).
\end{aligned} \tag{12}$$

In *direct problem* (12), it is required to determine the function $\eta(x,y,t)$ from the given function $q(x,y)$.

Now let $q(x,y)$ be unknown. Assume that the following additional information the solution to direct problem (12) is given

$$\eta(x,y,T) = f^{(2)}(x,y). \tag{13}$$

The inverse problem consists in determining the function $q(x,y)$ from (12), (13) and the given functions $f^{(1)}(x,y)$, $f^{(2)}(x,y)$, and $H(x,y)$.

3.1. The degree of ill-posedness of the inverse problem

As was shown in [9] Dirichlet problem (11) and inverse problem (12), (13) are ill-posed in terms of Hadamard.

Consider a simple case of $H(x,y) = g^{-1}$ and $b = \pi$. Take an odd extension of all functions in problem (12), (13) to the interval $(-\pi,\pi)$ with respect to the variable y. Then, taking the Fourier series expansion of the function

$$\eta(x,y,t) = \sum_{k\in\mathbb{N}} \eta_k(x,t) \sin ky$$

and of all the other functions, we obtain a sequence of the one-dimensional inverse problems ($k \in \mathbb{N}$)

$$\begin{aligned}
&\eta_{k,tt} = \eta_{k,xx} - k^2\eta_k, && x \in (0,\pi), \quad t \in (0,T), \\
&\eta_k(x,0) = f_k^{(1)}(x), \quad \eta_{k,t}(x,0) = q_k(x), && x \in (0,\pi), \\
&\eta_k(0,t) = \eta_k(\pi,t) = 0, && t \in (0,T);
\end{aligned} \tag{14}$$

$$\eta_k(x,T) = f_k^{(2)}(x). \tag{15}$$

Taking an odd extension of the functions $u_k(x,t)$, $f_k^{(1)}(x)$, $f_k^{(2)}(x)$, and $q_k(x)$ with respect to x to the interval $(-\pi,0)$, we expand them to Fourier series:

$$\eta_k(x,t) = \sum_{n\in\mathbb{N}} \eta_{k,n}(t) \sin nx,$$

and so on.

As a result, we have a sequence of the inverse problems

$$\begin{aligned} &\eta''_{k,n} + (n^2 + k^2)\eta_{k,n} = 0, \\ &\eta_{k,n}(0) = f^{(1)}_{k,n}, \quad \eta'_{k,n}(0) = q_{k,n}; \end{aligned} \tag{16}$$

$$\eta_{k,n}(T) = f^{(2)}_{k,n}. \tag{17}$$

Theorem 2 (uniqueness of the solution to inverse problem (16), (17)). *Assume that for all $k, n \in \mathbb{N}$, $m \in \mathbb{Z}$, the parameter $T \in (0, T_{\max})$ satisfies the condition $T \neq \frac{\pi m}{p_{k,n}}$ (for example, $T = \frac{r_1}{r_2}$ is a rational number in the interval $(0, T_{\max})$). If inverse problem (14), (15) has a solution in $C^1[0,\pi]$, then the solution is unique and its Fourier coefficients are given by the formula*

$$q_{k,n} = \frac{f^{(2)}_{k,n} - f^{(1)}_{k,n} \cos p_{k,n}T}{\sin p_{k,n}T} p_{k,n}.$$

We define an operator $A(k) : L_2[0,\pi] \to L_2[0,\pi]$ of problem (14), (15) as follows:

$$[A(k)q_k](x) = F_k(x) := \sum_{n\in\mathbb{N}} \big(f^{(2)}_{k,n} - f^{(1)}_{k,n} \cos p_{k,n}T\big) \sin nx.$$

Theorem 3 [10]. *Singular values of the operator $A(k)$ have the form*

$$\sigma_n(A(k)) = \frac{|\sin p_{k,n}T|}{p_{k,n}}, \quad n \in \mathbb{N}.$$

Figure 2 shows that a sequence of singular values of the operator $A(k)$ decreases as n increases. Moreover, the singular values decrease as k increases.

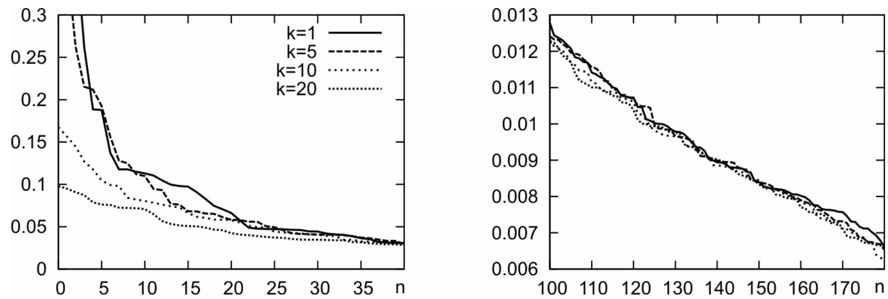

Figure 2. Singular values $\sigma_n(A(k))$ for $k = 1, 5, 10, 20$, $T = 3$: left—$n = 0, \ldots, 40$; right—$n = 100, \ldots, 180$

4. A numerical method for determining the amplitude of a wave edge in the shallow water approximation

In the open ocean, the wave height rarely exceeds one meter, and the wavelength (the distance between wave crests) may reach several hundreds km. These numbers are typical calculation domain sizes. It is necessary to solve the inverse problem for determination of the tsunami source parameters. The solution of the inverse problem is based on the solution of the wave propagation problem in the open ocean (the direct problem). The simulation of tsunami wave propagation in such scales is not an easy calculation task. A numerical algorithm, which makes possible to calculate the front amplitude of a wave coming to a given point (x_0, y_0) and the wave arrival time by solving this problem not in the entire domain, but only on a selected characteristic surface, is proposed.

We consider the Cauchy problem for shallow water equations (1):

$$\begin{aligned} &L\eta = 0, \quad (x,y) \in \mathbb{R}^2,\ t > 0; \\ &\eta(x,y,0) = q(x,y), \quad \eta_t(x,y,0) = 0, \quad (x,y) \in \mathbb{R}^2. \end{aligned} \tag{18}$$

Assume that the function $q(x,y)$ is represented in the form $q(x,y) = h(y)q_1(x)$. Here $h(y)$ is a smooth function, and the function $q_1(x)$ has the following form:

$$q_1(x) = \begin{cases} \tilde{q_1}(x), & x < -\epsilon, \\ x, & x \in (-\epsilon, 0), \\ 0, & x > 0, \end{cases}$$

where $\tilde{q_1}(x)$ is a smooth function, and $\epsilon > 0$ is a small parameter. Then the solution to problem (18) is a fundamental solution to the tsunami wave propagation problem.

Problem (18) can be reduced to the problem in a half-plane after some transformations [11, 12]:

$$\begin{aligned} &L\eta = 0, \quad x, y > 0,\ t > 0, \\ &\eta|_{t<0} \equiv 0, \quad \eta_x|_{x=0} = -\frac{1}{2} g(y)\delta(t) + \tilde{h}(y,t), \quad x, y > 0. \end{aligned} \tag{19}$$

Here $g(y) = c^2(0,y)h(y)$ and $\tilde{h}(y,t)$ are smooth functions.

Let $\alpha = y$ and $z = \tau(x,y)$ be new variables, where $\tau(x,y)$ is the solution to the following problem [13]:

$$\begin{aligned} &\tau_x^2 + \tau_y^2 = c^{-2}(x,y), \quad x > 0,\ y \in \mathbb{R}, \\ &\tau(0,y) = 0, \quad \tau_x > 0, \quad y \in \mathbb{R}. \end{aligned}$$

We change the variables

$$w(x,y,t) = v(z,\alpha,t), \quad b(z,\alpha) = c(x,y),$$

and represent the function $v(z, \alpha, t)$ as follows:

$$v(z, \alpha, t) = s(z, \alpha)\theta(t - z) + \overline{v}(z, \alpha, t), \quad t > z > 0,$$

where $s(z, \alpha)$ is the wave amplitude, $\overline{v}(z, \alpha, t)$ is a smooth function, and equate the coefficients to the delta-function $\delta(t - z)$ [12]:

$$\begin{aligned} &2s_z + 2b^2\tau_y s_\alpha + \left(b^2(\tau_{xx} + \tau_{yy}) + 2\frac{b_z}{b} + 2bb_\alpha\tau_y\right)s = 0, \quad z, \alpha > 0, \\ &s(0, \alpha) = \frac{g(\alpha)}{2\sqrt{b(0, \alpha)^{-2} - \tau_y^2}}, \quad \alpha > 0. \end{aligned} \tag{20}$$

Thus, the numerical algorithm constructed for solving the Cauchy problem for the wave equation makes possible to determine the wave front amplitude at a point of interest (x_0, y_0) in the spatial domain at any fixed time t_0.

In the one-dimensional case (when all functions depend on the variable x only) the solution of (20) has the form:

$$s(z) = s(0)\sqrt[4]{\frac{H(0)}{H(z)}}.$$

Note, that the amplitude of the wave edge is inversely proportional to the fourth root of the bottom topography function. Thus, the amplitude increases as a depth of the bottom decreases.

4.1. Numerical experiment

Let us apply the algorithm to the one-dimensional problem on the interval $[0, L]$, $L = 400$ km.

We define the bottom depth as $H(x) = H_{\max} - (\alpha + \beta x^2)$, $x \in [0, L]$. Here $\alpha = H_{\min} = 0.01$ km, $\beta = (H_{\max} - 2H_{\min})/L^2$, $H_{\max} = 4.5$ km. Then $c(x) = \sqrt{gH(x)} = \sqrt{g(H_{\max} - (\alpha + \beta x^2))}$. The amplitude of the initial wave perturbation is equal to 1 m. After substituting

$$z = \int_0^x \frac{d\lambda}{c(\lambda)} = \frac{\arcsin\sqrt{\beta x/(H_{\max} - H_{\min})}}{\sqrt{g\beta}},$$

we determine the amplitude of a wave edge in the entire domain $[0, L]$ (see Figure 3).

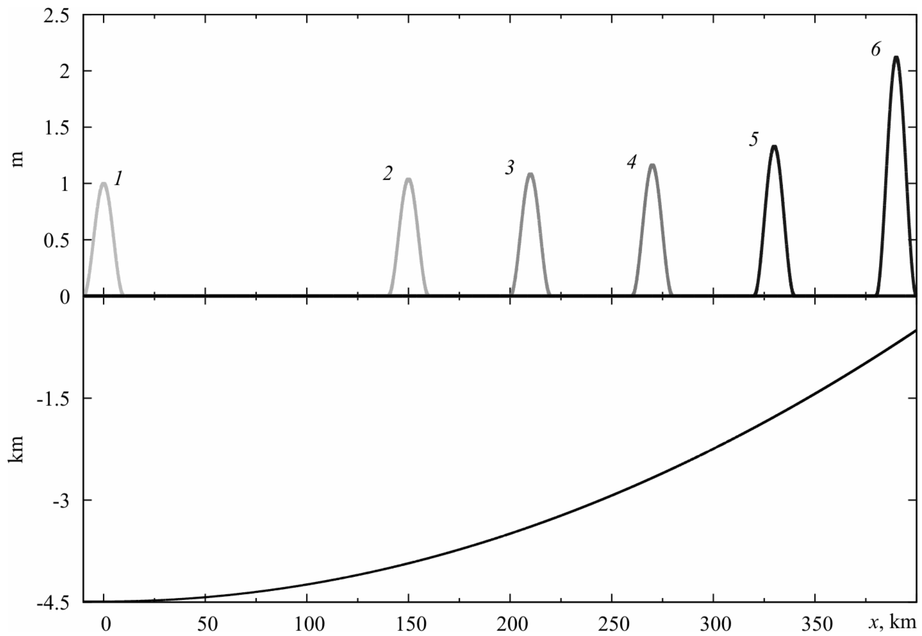

Figure 3. The bottom topography is described by the curve. The wave motion is described by the graphs: *1* — $t = 0$ min (initial position), *2* — $t = 12$ min, *3* — $t = 18$ min, *4* — $t = 24$ min, *5* — $t = 31$ min, *6* — $t = 43$ min.

5. Conclusion

We investigate three different inverse problems of determining a tsunami source using three different additional data: DART measurements, satellite wave-form images and seismic data. We describe the gradient-type and the SVD inverse problem solution. All these problems are linear. Combining all the above mentioned additional data, we use the termin "combined" not only for the sake of briefness, but keeping in mind that in practice specialists always deal with combined inverse problems. When trying to find a correct location for a possible tsunami source, one should take into account and combine all the results from seismic data, DART data, satellite data, and so on. It is evident that using a combination of three different types of data allows one to increase the stability and convergence of the numerical inverse problem solution.

References

[1] Pelinovskii E.N. Hydrodynamics of Tsunami Waves. — Nizhny Novgorod: Institute of Appl. Physics RAS, 1996 (In Russian).

[2] Gusiakov V.K. Excitation of Tsunami Waves and Oceanic Rayleigh Waves in a Seaquake // Mathematical Problems of Geophysics. — Novosibirsk: Computing

Center of the Siberian Branch of the USSR Acad. Sci., 1972. — No. 3. — P. 250–272.

[3] Bezhaev A.Yu., Lavrentiev M.M. (jr.), Marchuk An.G., Titov V.V. Determination of tsunami sources using deep ocean wave records // Bull. Novosibirsk Comp. Center. Ser. Math. Model. in Geoph. — Novosibirsk, 2006. — Iss. 7. — P. 53–63.

[4] Wei Y., Cheung K.F., Curtis G.D., McCreery C.S. Inverse algorithm for tsunami forecasts // J. Waterw. Port Coast. Ocean Eng. — 2003. — Vol. 129, — No. 3. — P. 60–69.

[5] Voronina T.A. Reconstruction of initial tsunami waveforms by a truncated SVD method // J. Inverse Ill-Posed Probl. — 2011. — Vol. 19, No. 4–5. — P. 615–629.

[6] Kabanikhin S., Hasanov A., Marinin I., et al. A numerical method for determining the amplitude of a wave edge in shallow water approximation // Appl. Num. Math., 2013 (to appear).

[7] Kabanikhin S.I. Inverse and Ill-Posed Problems: Theory and Applications. — Germany: de Gruyter, 2012.

[8] Okal E., Piatanesi A., Heinrich P. Tsunami detection by satellite altimetry // J. Geophys. Res. — 1999. — Vol. 104, No. B1.

[9] Kabanikhin S.I., Bektemesov M.A., Nurseitov D.B., et al. An optimization method in the Dirichlet problem for the wave equation // J. Inverse Ill-Posed Probl. — 2012. — Vol. 20, No. 2. — P. 193–211.

[10] Kabanikhin S.I., Krivorotko O.I. The numerical method of solving Dirichlet problem for the wave equation // J. Appl. Indust. Math. — 2012. — Vol. 15, No. 4. — P. 90–101 (In Russian).

[11] Vladimirov V.S. Equations of Mathematical Physics. — Moscow: Nauka, 1981 (In Russian).

[12] Kabanikhin S.I., Krivorotko O.I. A numerical method for determining the amplitude of a wave edge in shallow water approximation // Appl. Comp. Math. — 2013. — Vol. 12, No. 1. — P. 91–96.

[13] Kabanikhin S.I. Linear regularization of multidimensional inverse problems for hyperbolic equations. — Novosibirsk, 1989. — (Preprint / Sobolev Institute of Mathematics; 27) (In Russian).