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A NUMERICAL METHOD FOR DETERMINING THE AMPLITUDE OF A WAVE EDGE IN SHALLOW WATER APPROXIMATION

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Abstract. An algorithm to solve numerically the problem of determining the leading edge amplitude of a wave is constructed. The wave is generated by the initial condition $u(x, y, 0) = g(y) \cdot \delta(x)$. By the change of variables $z = \tau(x, y)$, $\alpha = y$, where τ is the solution to the eikonal equation

$$\tau_x^2 + \tau_y^2 = c^{-2}(x, y),$$

$c(x, y) = \sqrt{gH(x, y)}$, and $H(x, y)$ is the depth at point (x, y), the problem is reduced to solving of the first-order partial differential equation on the plane $t = z$. The algorithm makes it possible to calculate the front amplitude of a wave coming to the given point (x_0, y_0), as well as the wave arrival time.

Keywords: Shallow Water Equations, Eikonal Equation, Wave Front Amplitude, Numerical Calculations.

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1. Introduction

Strong earthquakes and related catastrophic tsunamis in the Pacific and Indian Oceans took place in the last decade. These events raised important problems of predicting tsunamis, determining the coastal wave parameters, and assessing the damage to the nearby cities. Any prediction of a tsunami event implies some errors (data errors produced by seismic and above-water sensors, calculation and model errors, etc.). As a result, the probability of a false prediction increases. Therefore, it is important to achieve high accuracy in tsunami prediction. An important problem here is to estimate the tsunami source (that is, earthquake) parameters. In paper [6] connection between the earthquake's parameters and the initial perturbation of water surface was shown. Therefore, the problem of determination of earthquake's parameters reduces to the problem of determination of the initial perturbation form of free surface [2].

In the open ocean, the wave height rarely exceeds one meter, and the wavelength (the distance between wave crests) may reach several hundred km [7]. These numbers are typical calculation domain sizes. It is necessary to solve the inverse problem for determination the tsunami source parameters. The solution of inverse problem based on the solution of problem of wave propogation in the open ocean (the direct problem). Simulation of tsunami wave propagation on such scales is not an easy calculation task. A numerical algorithm is proposed which makes it possible to calculate the front amplitude of a wave coming to a given point (x_0, y_0) and the wave arrival time by solving this problem not in the entire domain, but only on a selected characteristic surface.

In Section 2 the problem statement is formulated. In Section 3 the algorithm for determining the amplitude of a wave edge is proposed. In Section 4 the application of the algorithm to

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the quasi-2D case is described and in Section 5 the results of some numerical calculations are presented.

2. Problem statement

The motion of an incompressible liquid with a free surface under gravity can be described in shallow water approximation [7]. This approximation is based on the assumption that the liquid depth is small in comparison to the horizontal dimensions of the calculation domain. In this case, the vertical velocity component, versus the horizontal components, is small and can be ignored.

In a Cartesian system of coordinates, we write a Cauchy problem for the linear equations of shallow water theory in terms of the liquid flow components in dimensional form without allowance for the action of external forces, e.g. the Coriolis force and bottom friction

$$\begin{cases} \dfrac{\partial \eta}{\partial t} + \dfrac{\partial (Hu)}{\partial x} + \dfrac{\partial (Hv)}{\partial y} = 0, \\ \dfrac{\partial u}{\partial t} + g\dfrac{\partial \eta}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} + g\dfrac{\partial \eta}{\partial y} = 0. \end{cases} \qquad (x,y) \in \mathbb{R}^2,\ t > 0; \tag{1}$$

$$\eta|_{t=0} = \phi(x,y), \quad u|_{t=0} = 0, \quad v|_{t=0} = 0, \quad (x,y) \in \mathbb{R}^2. \tag{2}$$

Here $\eta(x,y,t)$ is the free surface, $H(x,y) > 0$ is a known function describing the bottom relief (bathymetry), $u(x,y,t)$ and $v(x,y,t)$ are the velocity field components in the Ox and Oy directions, respectively, and $g = 9,8\,\mathrm{m/s^2}$ is the acceleration of gravity.

The initial conditions (2) describe the initial water surface when a tsunami takes place as a result of a seaquake [1].

Differentiating the first equation in system (1), we proceed from the Cauchy problem for the first-order system of equations (1), (2) to the following Cauchy problem for the second-order hyperbolic equation

$$\begin{cases} L\eta \equiv \eta_{tt} - div\ \left(c^2(x,y) grad\, \eta\right) = 0, & (x,y) \in \mathbb{R}^2,\ t > 0; \\ \eta|_{t=0} = \phi(x,y), \quad \eta_t|_{t=0} = 0, & (x,y) \in \mathbb{R}^2. \end{cases} \tag{3}$$

Here $c(x,y) = \sqrt{gH(x,y)}$ is the propagation speed of a surface perturbation.

It is well-known [8] that there exists a generalized solution to problem (3), and it is unique.

Assume that the function $\phi(x,y)$ is represented in the form

$$\phi(x,y) = h(y) \cdot \phi_1(x). \tag{4}$$

Here $h(y)$ is a smooth function, and the function $\phi_1(x)$ has the following form

$$\phi_1(x) = \begin{cases} \tilde{\phi_1}(x), & x < -\epsilon, \\ x, & x \in (-\epsilon, 0), \\ 0, & x > 0, \end{cases}$$

where $\tilde{\phi_1}(x)$ is a smooth function, and $\epsilon > 0$ is a small parameter. Then a solution to problem (3) with a function $\phi(x,y)$ of form (4) is a fundamental solution to the problem of tsunami wave propagation.

Substituting $\eta_t(x,y,t) = w(x,y,t)$, we proceed from problem (3) to the problem

$$\begin{cases} Lw = 0, & (x,y) \in \mathbb{R}^2,\ t > 0; \\ w|_{t=0} = 0, \quad w_t|_{t=0} = \psi(x,y), & (x,y) \in \mathbb{R}^2. \end{cases} \tag{5}$$

Here the function $\psi(x,y)$ takes the following form

$$\psi(x,y) = \begin{cases} div\left(c^2(x,y)\, grad\left(h(y)\tilde{\phi_1}(x)\right)\right), & x < -\epsilon, \\ g(y)\cdot\delta(x) + \tilde{\psi}(x,y), & x \in (-\epsilon, 0), \\ 0, & x > 0, \end{cases} \tag{6}$$

where $g(y) = c^2(0,y)h(y)$ is a smooth function, $\delta(x)$ is the Dirac delta-function, and $\tilde{\psi}(x,y) = c(x,y)\left(2c_x(x,y)h(y) + xc(x,y)h''(y) + 2xc_y(x,y)h'(y)\right)$ is a smooth function.

In what follows, the Cauchy problem (5) will be the subject of our study.

It is well-known [8] that problem (5) with a function $\psi(x,y)$ of form (6) can be reduced to the following problem in a half-plane

$$\begin{cases} Lw = 0, & x, y > 0,\ t > 0; \\ w|_{t<0} = 0, & w_x|_{x=0} = -\frac{1}{2}g(y)\cdot\delta(t) + \tilde{h}(y,t). \end{cases} \tag{7}$$

Here $\tilde{h}(y,t)$ is a smooth function.

3. Eikonal transform

In this section, we described a numerical algorithm for solving problem (7), based on the approach proposed in [3].

Let $\alpha = y$ and $z = \tau(x,y)$ be new variables, where $\tau(x,y)$ is the solution to the following problem

$$\begin{cases} \tau_x^2 + \tau_y^2 = \dfrac{1}{c^2(x,y)}, & x > 0, \quad y \in \mathbb{R}; \\ \tau(0,y) = 0, & \tau_x > 0, \quad y \in \mathbb{R}. \end{cases} \tag{8}$$

Then, with the change of variables

$$v(z,\alpha,t) = w(x,y,t), \quad b(z,\alpha) = c(x,y), \tag{9}$$

problem (7) can be rewritten as follows:

$$\begin{cases} v_{tt} = v_{zz} + b^2 v_{\alpha\alpha} + 2b^2\tau_y v_{z\alpha} + \left(b^2(\tau_{xx} + \tau_{yy}) + 2\dfrac{b_z}{b} + 2bb_\alpha\tau_y\right) v_z + \\ \qquad + 2b(b_z\tau_y + b_\alpha)v_\alpha, \quad z, \alpha > 0,\ t > 0; \\ v|_{t<0} = 0, \quad v_z|_{z=0} = -\dfrac{g(\alpha)}{2\sqrt{\frac{1}{b^2(0,\alpha)} - \tau_y^2}}\delta(t) + \dfrac{\tilde{h}(\alpha,t)}{2\sqrt{\frac{1}{b^2(0,\alpha)} - \tau_y^2}}. \end{cases} \tag{10}$$

Remark 1. The first equation of system (8) is called the eikonal equation. It describes characteristic surfaces $t = \tau(x,y,x^0,y^0)$ having a cone at the fixed point (x^0,y^0) [7]. The point (x^0,y^0) (an earthquake epicenter) is assumed to be known. It will be omitted as a parameter, e.g. $\tau(x,y) = \tau(x,y,x^0,y^0)$.

Let the solution to problem (10) be represented as follows

$$v(z,\alpha,t) = s(z,\alpha)\cdot\theta(t-z) + \overline{v}(z,\alpha,t), \quad t > z > 0. \tag{11}$$

Here $\overline{v}(z,\alpha,t)$ is a smooth function.

Then, equating the coefficients at the delta-function $\delta(t-z)$, we obtain a problem for the wave amplitude $s(z,\alpha)$

$$\begin{cases} 2s_z + 2b^2\tau_y s_\alpha + \left(b^2(\tau_{xx}+\tau_{yy}) + 2\frac{b_z}{b} + 2bb_\alpha\tau_y\right)s = 0, & z,\alpha > 0; \\ s(0,\alpha) = \dfrac{g(\alpha)}{2\sqrt{\frac{1}{b^2(0,\alpha)} - \tau_y^2}}, & \alpha > 0. \end{cases} \tag{12}$$

Thus, the numerical algorithm constructed for solving the Cauchy problem for the wave equation makes it possible to determine the wave front amplitude at a point of interest (x_0, y_0) in the spatial domain at any fixed time t_0.

4. Application to quasi-2D problem

Suppose that functions $c(x,y) = c(x)$ and $\phi(x,y) = \phi(x)$ in problem (3) depend only on variable x. Assuming that $y \in [0, A]$, the velocity of wave propagation on the boundary is equal zero, e.g. $\eta_y\big|_{y=0} = \eta_y\big|_{y=A} = 0$, and using the substitution

$$w(x,t) = \int\limits_0^A \eta(x,y,t)\,dy,$$

get the following Cauchy problem

$$\begin{cases} w_{tt} = \left(c^2(x)w_x\right)_x, & x \in \mathbb{R},\ t > 0; \\ w\big|_{t=0} = 0, \quad w_t\big|_{t=0} = \delta(x), & x \in \mathbb{R}. \end{cases} \tag{13}$$

Similar to the two-dimensional case, we obtain an analogue of the problem (7) on the half-plane

$$\begin{cases} w_{tt} = \left(c^2(x)w_x\right)_x, & x > 0,\ t > 0; \\ w\big|_{t<0} = 0, & w_x\big|_{x=0} = -\frac{1}{2}\cdot\delta(t) + \tilde{h}(t). \end{cases}$$

Introducing new variable

$$z = \int\limits_0^x \frac{d\lambda}{c(\lambda)}$$

and new functions $v(z,t) = w(x,t)$, $b(z) = c(x)$, we obtain the following problem

$$\begin{cases} v_{tt} = v_{zz} + \frac{b_z}{b}v_z, \quad z > 0,\ t > 0; \\ v\big|_{t<0} = 0, \quad v_z\big|_{z=0} = -\frac{b(0)}{2}\delta(t) + \frac{b(0)}{2}\tilde{h}(t). \end{cases} \tag{14}$$

Representing the solution to problem (14) in the form $v(z,t) = s(z)\cdot\theta(z-t) + \tilde{v}(z,t)$, we arrive to the Cauchy problem of determining the amplitude of a wave edge

$$\begin{cases} 2s'(z) + \frac{b'(z)}{b(z)}s(z) = 0, \quad z > 0; \\ s(0) = \frac{b(0)}{2}. \end{cases} \tag{15}$$

The solution of (15) has the form

$$s(z) = s(0)\cdot\sqrt{\frac{b(0)}{b(z)}}. \tag{16}$$

Note, that the amplitude of the wave edge is inversely proportional to the velocity of propagation of the surface waves, that is related to the bottom topography. Thus, the amplitude increases as a depth of the bottom decreases. This explains the wave increasing at the shore (Fig.1).

5. Numerical experiment

Let us apply the algorithm to the problem (13) on the interval $[0, L]$, $L = 400$ km.

We consider a discrete domain $x_i = ih_x$, where $0 \le i \le N_x$. Here N_x is the number of points in $x \in (0, L)$.

We define the bottom depth as follows

$$H(x) = H_{max} - \left(\alpha + \beta x^2\right), \quad x \in [0, L].$$

Here $\alpha = H_{min} = 0.01$ km, $\beta = (H_{max} - H_{min})/L^2$, $H_{max} = 4.5$ km. Then $c(x) = \sqrt{gH(x)} = \sqrt{g\,(H_{max} - (\alpha + \beta x^2))}$. The amplitude of initial wave perturbation is equal to 1 m. After substituting

$$z = \int\limits_0^x \frac{d\lambda}{c(\lambda)} = \frac{\arcsin \sqrt{\frac{\beta}{H_{max}-H_{min}}}x}{\sqrt{g}\cdot\sqrt{\beta}},$$

we solve the problem (15) and determine the amplitude of a wave edge in the entire domain $[0, L]$:

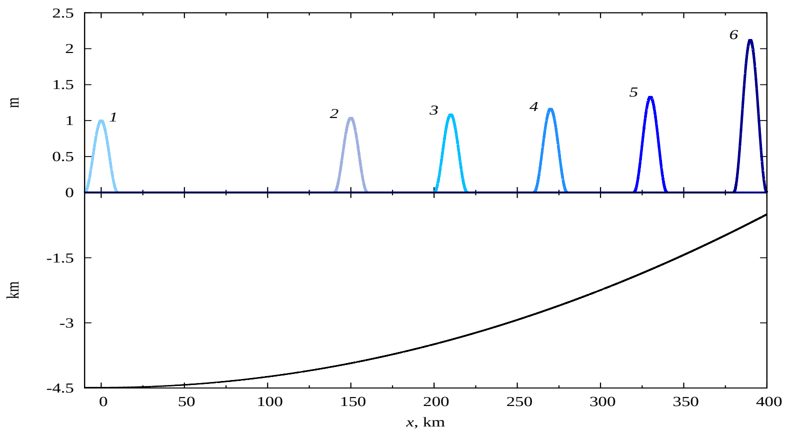

Figure 1. The bottom topography is described by the curve. All sizes are given in km. The wave motion is described by graphs: (*1*) - $t = 0$ min (initial position), (*2*) – $t = 12$ min, (*3*) – $t = 18$ min, (*4*) – $t = 24$ min, (*5*) – $t = 31$ min, (*6*) – $t = 43$ min.

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7. Conclusion

In the paper a numerical solution is given for the problem of determining the leading edge amplitude of a wave generated by some initial condition. The problem is reduced to solving of the first order partial differential equation. Then the algorithm for determining the amplitude of a wave edge is proposed. Finally the application of the algorithm to the quasi-2D case is described, and the results of some numerical calculations are presented.

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